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A NEW TRIGONOMETRY FOR SCHOOLS

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P. 88

BY

W. G. BORCHARDT, M.A., B.Sc.
ASSISTANT MASTER AT CHELTENHAM COLLEGE
FORMERLY SCHOLAR OF ST JOHN'S COLLEGE, CAMBRIDGE

AND

THE REV. A. D. PERROTT, M.A.
FORMERLY HEADMASTER OF COVENTRY GRAMMAR SCHOOL
FORMERLY SCHOLAR OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE

Ch. [illegible] Rao

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S. Ramalingam

PREFACE.

THE recent changes in the methods of teaching Elementary Mathematics (so largely due to the genius of Prof. Perry) have considerably affected Plane Trigonometry. Students are expected to have a good *practical* knowledge of the subject, while great skill in the solution of artificial problems and identities has ceased to be regarded as the aim and object of the subject.

This book has been written with a view to these changes and to supply the need felt for a School Trigonometry based on the use of *Four Figure Logarithms*, in which Logarithms, the Solution of Triangles and the more practical parts of the subject are introduced as early as possible. For this reason the expansions of $\sin(A+B)$, etc. and harder identities are deferred until after the Solution of Triangles, Heights and Distances, etc.

Seeing that incommensurable quantities are now omitted in Elementary Geometry and consequently no difficulty is found with the various theorems relating to arcs and sectors of circles, it has been thought advisable to place the Circular Measurement of Angles immediately after the measurement in degrees, etc.

Graphical Methods and *Squared Paper* are largely employed in the approximation to trigonometrical ratios of a given angle, in finding angles from given ratios, in the variations of trigonometrical expressions and logarithms.

Students are advised always to *check* their results in the Solution of Triangles, Heights and Distances, etc., by drawing figures to scale.

The more *theoretical* parts are treated with fulness for the benefit of those intending to proceed to higher branches of mathematics.

Part I includes Solution of Triangles, Heights and Distances, and Functions of Compound Angles, and is sufficient for the Oxford and Cambridge Junior Local, Mathematics I of the Woolwich and Sandhurst Examination, etc. It contains over 1200 examples.

Part II contains chapters on De Moivre's Theorem, the Exponential Theorem and the expansion of $\sin\theta$ and $\cos\theta$ in terms of θ, etc.

Considerable care has been given to the selection of examples, many of which are taken from recent Army and Navy Entrance and the various Cambridge Examinations.

An appendix on the *Slide Rule* will be found useful for students preparing for the Entrance Examinations to Woolwich and Sandhurst.

It is hoped that the sets of *Test Papers*, which have been very carefully graduated to fit in with the sequence of chapters in the book, will prove useful for revision. Harder questions will be found in the Miscellaneous Examples.

The examples have all been verified from the proof sheets and it is hoped that very few errors remain; in the use of four figure tables, answers vary slightly according to the precise method of working; *e.g.* log 4 is not exactly the same as 2 log 2; such variations occur chiefly in solving triangles when there are several formulae applicable; the authors have in many cases indicated which formulae should be used to obtain the answers in the book.

The authors wish to express their gratitude for many suggestions received from Mr T. Hyett of Cheltenham College.

CHELTENHAM COLLEGE, *September* 1894.

PREFACE TO THE EIGHTH EDITION.

THE present edition contains a new Appendix on Projection, as well as other alterations to the original edition.

The authors take the opportunity of thanking Mr R. C. Chevalier, of Manchester Grammar School, for much valuable advice and criticism. In particular, the new proofs given on pages 117 *a*, 117 *b*, Art. 84, and the Alternative Proofs in the Appendix on Projection are due to him.

Mr J. M. Child has independently evolved proofs similar to those on page 117 *b* and they are published in Barnard and Child's 'New Geometry,' while Mr W. Clark of Uddington Grammar School kindly sent the authors a proof similar to that in Art. 84.

September, 1912.

H Ramalingam
Junior Inter

CONTENTS.

PART I.

PART II.

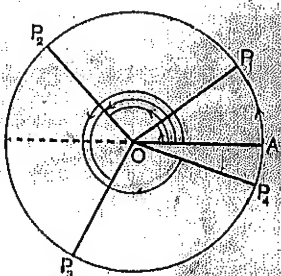
CHAPTER I.

MEASUREMENT OF ANGLES.

1. THE student is expected to be familiar with the definitions and explanations of an angle and a right angle as found in ordinary geometrical text-books.

2. The following slight extensions occur in Trigonometry.

Let a line OP revolve about the point O so that the point P traces out a circle, the direction of rotation being that shown in the figure, *i.e.* counter-clockwise. Let OA be the position from which OP starts to revolve.



When in the position OP_1 the angle described by OP may be either P_1OA or P_1OA + any multiple of 4 right angles, for OP might revolve any number of times in a complete circle before finally taking up the position OP_1.

In the positions OP_2, OP_3 and OP_4 the angles are as shown in the figure or these angles + any multiple of 4 right angles.

If OP revolved in the opposite direction to OP_4, then the angle P_4OA would be a negative angle and *e.g.* would be written $-30°$.

3. The geometrical unit is a right angle, but in Trigonometry this is subdivided.

1st method.

a right angle is divided into 90 equal parts called		**Degrees**
a degree „ „ „ 60 „ „ „		**Minutes**
a minute „ „ „ 60 „ „ „		**Seconds**

Thus 1 right angle $= 90^\circ$ (degrees)

$1^\circ = 60'$ (minutes)

$1' = 60''$ (seconds).

This is called the English or Sexagesimal method and in practice is universally employed.

2nd method.

a right angle is divided into 100 equal parts called	Grades
a grade „ „ „ „ „ „ „	Minutes
a minute „ „ „ „ „ „ „	Seconds

Thus 1 right angle $= 100^g$ (grades)

1 grade $= 100^{`}$ (minutes)

1 minute $= 100^{``}$ (seconds).

This is called the French or Centesimal method and is never employed in practice.

Rule. To convert Sexagesimal into Centesimal measure or vice versa, express the angle as a decimal of a right angle, then reduce to the new measure. (See examples 5 and 6.)

TYPICAL EXAMPLES.

Ex. 1. Reduce $21^\circ\ 13'\ 5''$ to seconds.

$$\begin{array}{rl} 21^\circ\ 13'\ 5'' & \\ 60 & \\ \hline 1273 & \text{minutes} \\ 60 & \\ \hline 76385 & \text{seconds.} \end{array}$$

Ex. 2. Reduce 82097″ to degrees, etc.

$$\begin{array}{r|l} 60 & 82097'' \\ \hline 60 & 1368'\ 17'' \\ \hline & 22^\circ\ 48'\ 17''. \end{array}$$

Ex. 3. Reduce $21^g\ 13^{`}\ 5^{``}$ to centesimal seconds.

Since the system is a decimal system the answer can be written down at sight:

$$21^g\ 13^{`}\ 5^{``} = 21^g\ 13^{`}\ 05^{``} = 211305^{``}.$$

Ex. 4. Reduce $320827^{``}$ to Grades, etc.

$$320827^{``} = 32^g\ 08^{`}\ 27^{``} = 32^g\ 8^{`}\ 27^{``}.$$

Ex. 5. Convert 64° 11′ 33″ to Centesimal measure.

$$\begin{array}{r|l} 60 & 33'' \\ \hline 60 & 11'{\cdot}55 \\ \hline 90 & 64^\circ{\cdot}1925 \\ \hline & {\cdot}71325 \text{ right angle.} \end{array}$$

Ans. $71^g\ 32^{`}\ 50^{``}$.

Ex. 6. Convert $64^g\ 11^{`}\ 33^{``}$ to Sexagesimal measure.

The angle = ·641133 of a right angle

$$\begin{array}{l} 90 \\ \hline 57^\circ{\cdot}70197 \\ 60 \\ \hline 42'{\cdot}1182 \\ 60 \\ \hline 7''{\cdot}092. \end{array}$$

Ans. 57° 42′ 7″·092.

EXAMPLES I.

1. Convert to minutes

$5^\circ\ 12'$; $60^\circ\ 28'$; $132^\circ\ 52'$.

2. Convert to seconds

$12^\circ\ 13'\ 8''$; $48^\circ\ 37'\ 29''$; $105^\circ\ 24'\ 31''$.

3. Convert to centesimal minutes

$5^g\ 12^`$; $60^g\ 9^`$; $132^g\ 98^`$.

4. Convert to centesimal seconds

$12^g\ 13^`\ 8^{``}$; $54^g\ 92^`\ 94^{``}$; $112^g\ 2^`\ 4^{``}$.

Convert to centesimal measure

5. $6^\circ\ 18'$; $18^\circ\ 27'$; $57^\circ\ 19'\cdot 8$.

6. $30^\circ\ 46'\ 48''$; $63^\circ\ 39'\ 35''$; $35^\circ\ 10'\ 39''$.

Convert to sexagesimal measure

7. $5^g\ 19^`$; $17^g\ 23^`$; $56^g\ 22^`$.

8. $31^g\ 47^`\ 41^{``}$; $64^g\ 3^`\ 5^{``}$; $79^g\ 91^`\ 7^{``}$.

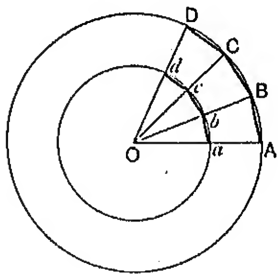
CIRCULAR MEASURE.

4. Theorem.

In all circles the ratio

$$\frac{\textit{circumference}}{\textit{diameter}} \textit{ is always the same.}$$

Take any two circles radii R and r and place them so that they have the same centre O. Divide the circles into n equal sectors by the lines OaA, ObB, OcC, etc. Join ab, bc, cd... and AB, BC, CD..., we then have 2 regular polygons of n sides inscribed in the circles.



$$\therefore \frac{\text{Perimeter of outer polygon}}{\text{Perimeter of inner polygon}} = \frac{n\,.\,\mathrm{AB}}{n\,.\,ab} = \frac{\mathrm{AB}}{ab} = \frac{\mathrm{R}}{r}$$

since ab and AB are parallel.

If n becomes indefinitely great and consequently AB, BC, CD... and ab, bc, cd... indefinitely small, the perimeters of the polygons become the circumferences of the circles;

$$\therefore \frac{\text{circumference of outer circle}}{\text{circumference of inner circle}} = \frac{\mathrm{R}}{r} = \frac{\text{diam. of outer circle}}{\text{diam. of inner circle}}.$$

$$\text{Hence } \frac{\text{circumference of any circle}}{\text{diameter of that circle}} = \text{a constant ratio.}$$

This constant ratio is an incommensurable number and is denoted by π.

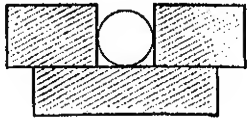
$\therefore$ **circumference** of a circle $= \pi\,.\,\mathrm{D} = \mathbf{2\pi r}$.

N.B. $\pi = 3{\cdot}1416$ approx.

$= \frac{22}{7}$ roughly.

5. The above result may be experimentally verified thus:

(i) Find the diameters of various coins by placing them between three rectangular blocks as in the figure.



(ii) Find the circumferences

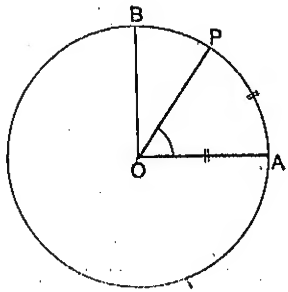
(*a*) with cotton

or (*b*) by making a small blot of ink on the rim of the coin, rolling it down a piece of cardboard and measuring the distance between two consecutive blots.

6. **Theorem.**

In any circle centre O *suppose an arc* AP *taken whose length = the radius.*

The angle POA *is constant for all circles.*



Draw OB perpendicular to OA.

Then

$$\frac{\hat{POA}}{\hat{BOA}} = \frac{\text{arc PA}}{\text{arc BA}}$$

$$= \frac{\text{radius OA}}{\text{one quarter of circumference}} = \frac{r}{\frac{2\pi r}{4}}$$

$$= \frac{2}{\pi};$$

$$\therefore\ \hat{POA} = \frac{2}{\pi} \text{ of a right angle} = \text{a constant}$$

$$= \frac{180^\circ}{\pi} = \frac{180^\circ}{3{\cdot}1416} = 57^\circ\ 17'\ 44'' \text{ approx.}$$

This constant angle is called a **Radian**.

Thus $$1 \text{ Radian} = \frac{2}{\pi} \text{ of a right angle,}$$

$$\therefore \pi \text{ radians} = 2 \text{ right angles}$$
$$= 180°.$$

DEF. The angle subtended at the centre of a circle by an arc equal in length to the radius is called a *Radian*.

DEF. The *Circular Measure* of an angle is the number of radians it contains.

N.B. The ratio of one angle to another is the same whatever the units used.

$$\therefore \frac{\text{no. of degrees in an angle}}{90} = \frac{\text{no. of grades in same angle}}{100}$$
$$= \frac{\text{no. of radians}}{\frac{\pi}{2}}.$$

Ex. 1. Convert $\frac{\pi}{12}$ radians to sexagesimal measure.

$$\pi \text{ radians} = 180°,$$
$$\therefore \frac{\pi}{12} \text{ radians} = \frac{180°}{12} = 15°.$$

Ex. 2. Convert 1·76 radians to sexagesimal measure.

$$\pi \text{ radians} = 180°,$$
$$\therefore 1 \text{ radian} = \frac{180}{\pi},$$
$$\therefore 1{\cdot}76 \text{ radians} = \frac{180°}{\pi} \times 1{\cdot}76$$
$$= \frac{7 \times 180}{22} \times 1{\cdot}76 \text{ approx.}$$
$$= 100°{\cdot}8 \text{ approx.}$$

For greater accuracy π should be taken as 3·1416.

Ex. 3. Convert 64° 11′ 33″ to circular measure.

$$\begin{array}{r|l} 60 & 33'' \\ \hline 60 & 11'{\cdot}55 \\ \hline & 64°{\cdot}1925. \end{array}$$

Now $$180° = \pi \text{ radians,}$$

$$\therefore\ 64°\cdot 1925 = \frac{\pi}{180} \times 64\cdot 1925 \text{ radians}$$

$$= \frac{3\cdot 1416}{180} \times 64\cdot 1925 \text{ radians approx.}$$

$$= 1\cdot 1204 \text{ radians approx.}$$

$$\left[\text{or more roughly} = \frac{22}{7 \times 180} \times 64\cdot 1925 \text{ radians}\right.$$

$$\left. = 1\cdot 1208 \text{ radians}\right].$$

EXAMPLES II.

Express in degrees, using $\pi = \frac{22}{7}$:

1.	$\frac{\pi}{2}$ radians,	9.	$\frac{2\pi}{3}$ radians,
2.	$\frac{\pi}{3}$,,	10.	$\frac{3\pi}{4}$,,
3.	$\frac{\pi}{4}$,,	11.	$\frac{5\pi}{24}$,,
4.	$\frac{\pi}{5}$,,	12.	8·8 ,,
5.	$\frac{\pi}{6}$,,	13.	1·65 ,,
6.	$\frac{\pi}{7}$,,	14.	0·22 ,,
7.	$\frac{\pi}{8}$,,	15.	1·1 ,,
8.	$\frac{\pi}{9}$,,	16.	0·066 ,,

Express in circular measure as fractions of π:

17.	15°.	18.	18°.	19.	30°.	20.	36°.
21.	45°.	22.	54°.	23.	60°.	24.	75°.
25.	90°.	26.	120°.	27.	135°.	28.	180°.

Convert to circular measure, using $\pi = 3 \cdot 1416$ and working to 4 places:

29. $5^\circ\ 12'$; $17^\circ\ 33'$; $82^\circ\ 39'$.

30. $4^\circ\ 2'$; $19^\circ\ 24'$; $78^\circ\ 29'$.

31. Express in degrees and in radians the angle in a regular figure of 3, 4, 5, 6, or 8 sides.

32. Two angles are such that their difference is 20° and their sum $1\frac{1}{2}$ radians, find the angles in degrees, and radians $(\pi = \frac{22}{7})$.

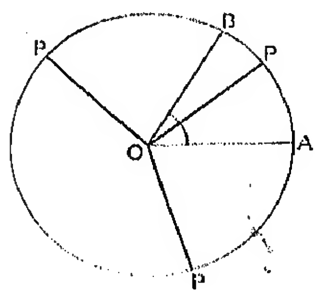
33. In a triangle one angle is 30° and another $\frac{\pi}{4}$, find the third angle in degrees, and radians $(\pi = \frac{22}{7})$.

34. An angle contains x sexagesimal minutes or y radians, find the ratio $x : y$, and hence state the multiplier necessary to convert radians into sexagesimal minutes; use $\pi = \frac{22}{7}$.

7. Theorem.

The circular measure of an angle

$$= \frac{\text{The arc which the angle subtends when at the centre of any circle}}{\text{The radius of that circle}}.$$



Let AOP be any angle and AOB a radian.

Then circular measure of $A\hat{O}P$

$$= \frac{A\hat{O}P}{A\hat{O}B} \text{ (def. Art. 6)}$$

$$= \frac{\text{arc } AP}{\text{arc } AB}$$

$$= \frac{\text{arc } AP}{\text{radius}}.$$

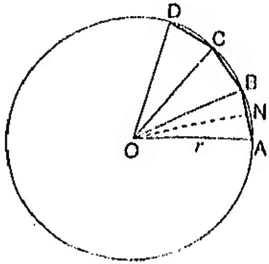
Hence if an arc of any circle radius r subtends an angle θ radians at the centre

$$\textbf{arc} = \mathbf{r}\boldsymbol{\theta}.$$

This agrees with the result obtained in Art. 4, for putting $\theta = 2\pi$

$$\text{circumference} = r \,.\, 2\pi = 2\pi r.$$

8. To find the area of a circle and of the sector of a circle.



Inscribe a regular polygon ABCD... in the circle. Draw ON perpendicular to AB.

Area of the polygon

$= \triangle AOB + \triangle BOC + \ldots\ldots$

$= \frac{1}{2} . ON . (AB + BC + \ldots\ldots)$

$= \frac{1}{2} . ON$. perimeter of polygon.

When the number of the sides is indefinitely increased

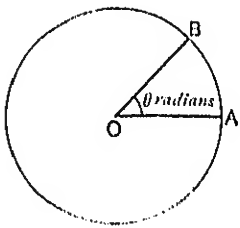
ON becomes the radius,

the perimeter becomes the circumference,

the area of polygon becomes the area of the circle,

$\therefore$ **area of circle** $= \frac{1}{2} . r . 2\pi r$

$= \boldsymbol{\pi r^2}$.



9. If AOB is a sector containing an angle θ radians,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi},$$

$$\therefore \frac{\text{area of sector}}{\pi r^2} = \frac{\theta}{2\pi};$$

$\therefore$ **area of sector** $= \frac{1}{2} \mathbf{r}^2\theta$.

Ex. 1. Calculate the number of degrees subtended at the centre of a circle of radius 6 centimetres by an arc 9 centimetres long.

$$\text{arc} = r\theta,$$

$$\therefore 9 = 6\theta,$$

or $$\theta = \tfrac{3}{2}.$$

Now $$\pi \text{ radians} = 180^\circ,$$

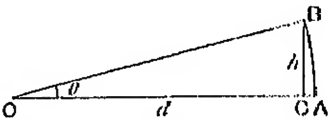
$$\therefore \theta \text{ radians} = \frac{180}{\pi} \times \frac{3^\circ}{2}$$

$$= \frac{180 \times 3 \times 7^\circ}{22 \times 2}$$

$$= 85\tfrac{10}{11}^\circ.$$

Ex. 2. Show that if an object of height h at distance d from the observer subtends a small angle A degrees at his position, then roughly $h = \frac{Ad}{57 \cdot 3}$.

Use this to find the height of a tower which subtends an angle of 9° at a point 170 yards away.



Let BC be the object and O the observer.

Draw a circle with centre O and radius OB meeting OC produced in A.

If h is small in comparison with d, then C is nearly coincident with A.

$$h = \text{arc AB (approx.)}$$
$$= \text{OA} . \theta = d\theta \text{ (approx.)}.$$

Now
$$\text{angle BOA} = \text{A}^\circ = \frac{180\,\theta}{\pi},$$

$$\therefore h = \frac{d\text{A}\pi}{180}$$
$$= \frac{22d\text{A}}{7 \times 180}$$
$$= \frac{\text{A}d}{57 \cdot 3} \text{ (approx.)};$$

$$\therefore \text{height of tower} = \frac{9 \times 170}{57 \cdot 3} \text{ yds.} = 26 \cdot 7 \text{ yds. (approx.)}.$$

EXAMPLES III.

[Assume $\pi = \frac{22}{7}$.]

1. Find the circumference of a circle whose radius is 5 cms.

2. If the circumference of a circle measures 50 cms., what is the length of the radius?

3. A wheel makes 20 revolutions per second, how long will it take to turn through 10° and 10 radians?

4. A water-tube boiler has 350 tubes of 2·5 inches internal diameter, and the length of each tube is 8 feet. Find the total heating surface (*i.e.* interior surface) of the tubes in square feet. (Area of curved surface of cylinder $= 2\pi rl$, where r is the radius and l the length.)

5. Find the circular measure of an angle subtended at the centre of a circle of radius 4 inches by an arc 13 inches long.

6. Calculate the number of degrees subtended at the centre of a circle of radius 5 centimetres by an arc 9 centimetres long.

7. An angle whose circular measure is ·52 is subtended at the centre of a circle by an arc 5 inches long; what is the radius?

8. Find the radius of a circle in which an arc of 8 inches subtends an angle of 25° at the centre.

9. What is the length of the arc of a circle of radius 2 metres which subtends an angle of 32° at the centre?

10. Find the length of the arc of a circle which subtends an angle whose circular measure is ·635 at the centre of a circle of radius 5 centimetres.

11. Assuming that the radius of the earth is 4000 miles, find the distance on the surface between two places on the same meridian the difference of whose latitude is 22°. Answer to the nearest mile.

12. A pendulum 8 feet long oscillates through an angle of 9°; what is the length of the path its extremity describes between the extreme positions?

13. If the sun is 92,000,000 miles away and its diameter subtends an angle of 32′ at the earth, find roughly the diameter of the sun to the nearest 100 miles.

14. Given that the moon's mean angular diameter is 31′ and that it is 240,000 miles away, find its actual diameter to the nearest mile.

15. If Ben Nevis, which is 4400 feet high, subtends an angle of 8° at Banavie, what is approximately its distance away?

16. Given that $\dfrac{\text{distance of Mars from Sun}}{\text{distance of Earth from Sun}} = 1{\cdot}52$, that the periodic time of the Earth is 365 days and of Mars 687 days, find the ratio of their speeds, assuming that they describe circular orbits.

17. If Neptune describes a circular orbit of radius 27×10^8 miles round the sun in 165 years, find the number of miles he goes per hour. Answer to the nearest mile.

---

18. Find the area of a circle whose radius is 3 centimetres.

19. The area of a circle is 154 sq. inches, find the radius.

20. Find the area of the sector of a circle whose radius is 5 feet and angle 25°.

21. Find the area of the sector of a circle when the radius is 4 feet, and the angle contains 1·5 radians.

22. The area of a circle is 154 sq. metres; find the length of the arc which subtends an angle of 50° at the centre.

23. The circumference of a circle is 66 inches; find the area.

24. The perimeter of a semicircle is 10 feet; find the area of a sector subtending an angle of 45° at the centre.

25. Find the area of a sector of a circle which has an angle of 2 radians and an arc of 10 inches.

26. A man standing beside one milestone on a straight road, observes that the foot of the next milestone is on a level with his eyes, and that its height subtends an angle of 2′ 55″. Find the approximate height of the milestone.

27. The sun is 93 million miles distant, and subtends an angle of ·00932 radians. Find its diameter.

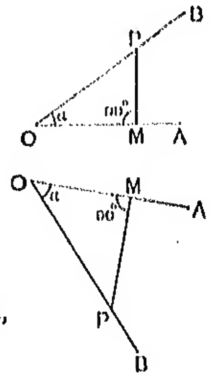
28. A circular wire of 3 inches radius is cut, and then bent so as to lie along the circumference of a hoop whose radius is 4 feet. Find the angle which it subtends at the centre of the hoop.

29. Four equal circles are placed so that each one touches two others: if the area between them is $\frac{150}{847}$ sq. inches, what is the length of the common radius?

CHAPTER II.

TRIGONOMETRICAL RATIOS.

10. IN this chapter we shall consider *acute angles* only. The extension to other angles will be given in Chapter VI.



Let AOB be an angle; if a point P be taken in one of the bounding lines of the angle and from it a perpendicular PM be drawn to the other bounding line (produced if necessary), then in the right-angled triangle so formed,

$$\frac{MP}{OP}\left(=\frac{\text{opposite side}}{\text{hypotenuse}}\right)=\text{sine of AOB}=\mathbf{sin}\ \alpha,$$

$$\frac{OM}{OP}\left(=\frac{\text{adjacent side}}{\text{hypotenuse}}\right)=\text{cosine of AOB}=\mathbf{cos}\ \alpha,$$

$$\frac{MP}{OM}\left(=\frac{\text{opposite side}}{\text{adjacent side}}\right)=\text{tangent of AOB}=\mathbf{tan}\ \alpha,$$

$$\frac{OP}{MP}\left(=\frac{\text{hypotenuse}}{\text{opposite side}}\right)=\text{cosecant of AOB}=\mathbf{cosec}\ \alpha,$$

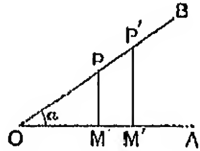
$$\frac{OP}{OM}\left(=\frac{\text{hypotenuse}}{\text{adjacent side}}\right)=\text{secant of AOB}=\mathbf{sec}\ \alpha,$$

$$\frac{OM}{MP}\left(=\frac{\text{adjacent side}}{\text{opposite side}}\right)=\text{cotangent of AOB}=\mathbf{cot}\ \alpha.$$

$$1-\sin\alpha=\text{coversed sine of }\alpha=\text{covers}\ \alpha,$$

$$1-\cos\alpha=\text{versed sine of }\alpha=\text{vers}\ \alpha.$$

These ratios are independent of the position of the
; for if any other point P′ be
n OB



$$\frac{M'P'}{OP'} = \frac{MP}{OP},$$

$$\frac{OM'}{OP'} = \frac{OM}{OP},$$

$$\frac{M'P'}{OM'} = \frac{MP}{OM}, \text{ etc.}$$

e triangles OPM and OP′M′ are similar.

above results may be verified by actual measurement.

Since the hypotenuse is the greatest side of a
ngled triangle, it follows that

$$\sin\alpha \ngtr 1,$$
$$\cos\alpha \ngtr 1,$$
$$\operatorname{cosec}\alpha \nless 1,$$
$$\sec\alpha \nless 1.$$

EXAMPLES IV.

If the two sides AC and CB of a right-angled triangle are
respectively, find the values of tan B, sec A, cosec A.

Given that one side BC of a right-angled triangle is 2 and
otenuse AB is $\sqrt{13}$, find the values of AC, cot A, sin B.

If the hypotenuse AB of a right-angled triangle is 16
tres, and the side BC is 5 centimetres, find the values of
s B, tan A, tan B, sin A, sin B.

t connection do you notice between sin A and cos B?

4. The two sides BC and AC of a right-angled triangle are 5 and 12 inches respectively. Find the values of the hypotenuse AB, tan B, cot A, sec B, cos A.

What connection do you notice between the tangent of an angle and the cotangent of the complement? (See Art. 27).

5. The sides AB, BC, CA of a right-angled triangle are 17, 15, 8 respectively; the sides PQ, QR, RP of another right-angled triangle are 61, 11, 60 respectively.

Find the values of $\sin^2 A + \cos^2 A$ and $\sin^2 P + \cos^2 P$. What do these results suggest?

6. The hypotenuse AB of a right-angled triangle is 41 centimetres and the side AC is 9 centimetres. Write down the values of sin B, tan B, cosec A, cos B.

What connection do you notice between sin B, cos B, tan B?

7. A church tower is 300 feet high; what is the sine of the angle it subtends at a point on the ground 400 ft. away from the foot of the tower?

8. A boat is 1000 yards away from a cliff 350 feet high; what are the sine and tangent of the angle which the cliff subtends at the boat?

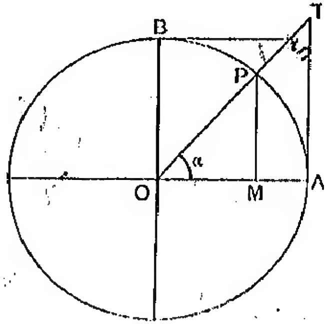
9. ABCD is a quadrilateral in which AB is at right angles to BC, and the diagonal AC is perpendicular to AD.

$$AB = 3, \quad BC = 4, \quad AD = 7.$$

Find the values of sin BAC, tan BCA, cot ACD, sec ADC.

10. The sides BC, CA of a right-angled triangle are 3 and 7 respectively. Find the values of tan A, sec A, cosec B, sin A, $\sqrt{1 + \tan^2 A}$. What trigonometrical ratio has the same value as the last expression?

13. Geometrical constructions for trigonometrical ratios with given angles.



The old method of constructing the trigonometrical ratios was to draw the given angle POA at the centre of a circle of unit radius, drop a perpendicular PM, and draw the tangents AT, Bt to the circle at A and B meeting OP or OP produced in T and t.

Then MP $=\sin\alpha$,

OM $=\cos\alpha$,

AT $=\tan\alpha$,

Ot $=$ cosec OtB $=$ cosec α,

OT $=\sec\alpha$,

Bt $=$ cot OtB $=\cot\alpha$.

Thus if the given angle is 50°, it is convenient to make OP $=10$ units, then measuring the lines given above and dividing by 10, we obtain

$\sin 50° = \cdot 77$, $\quad$ cosec $50° = 1\cdot 31$,

$\cos 50° = \cdot 64$, $\quad$ $\sec 50° = 1\cdot 56$,

$\tan 50° = 1\cdot 19$, $\quad$ $\cot 50° = \cdot 84$.

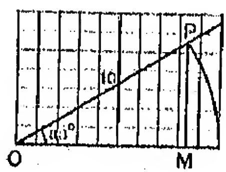
14. An alternative method is to calculate each ratio separately.

Ex. 1. Draw an angle of 33° and by a geometrical construction find the value of sin 33°.

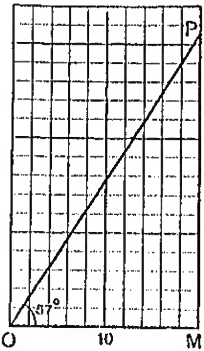
With a protractor draw the angle

POM $=33°$.

Measure OP $=10$ units and draw PM perpendicular to OM. Then



$$\sin 33° = \frac{MP}{OP} = \frac{5\cdot 4}{10} = \cdot 54.$$



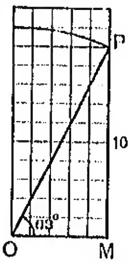
Ex. 2. Draw an angle of 57° and find by measurement the value of tan 57°.

By means of a protractor, construct the angle POM = 57°.

Measure OM = 10 units and erect a perpendicular MP from M.

$$\tan 57° = \frac{MP}{OM} = \frac{15\cdot4}{10} = 1\cdot54.$$

Ex. 3. Draw an angle of 63° and find by measurement the value of cosec 63°.



Draw a line PM = 10 units and construct the angle MPO = 27°. From M draw a perpendicular MO to meet PO at O.

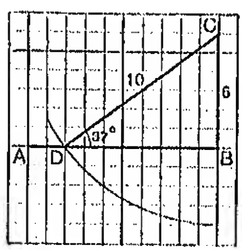
Then

$$\angle POM = 90° - OPM = 90° - 27° = 63°,$$

$$\text{cosec } 63° = \frac{OP}{MP} = \frac{11\cdot2}{10} = 1\cdot12.$$

15. Geometrical constructions for angles with given ratios.

Ex. 1. Draw an angle whose sine is $\frac{3}{5}$.



Take a line AB, and from B measure off BC = 6 divisions. With centre C and radius 10 divisions, draw an arc of a circle cutting AB in D.

Then

$$CD = \text{radius} = 10 \text{ divisions}.$$

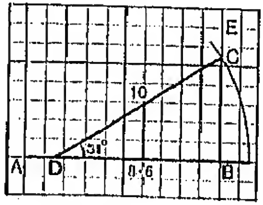
Since $$\sin CDB = \frac{BC}{DC} = \frac{6}{10} = \frac{3}{5},$$

∴ CDB is the angle required.

If measured with a protractor

$$\angle CDB = 37° \text{ (nearly)}.$$

Ex. 2. Construct an angle whose cosine is ·86.



From B measure BD = 8·6 divisions.

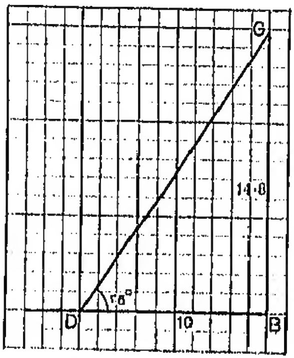
With centre D and radius 10 divisions describe an arc cutting BE at C.

CDB is required angle = 31° (nearly).

Ex. 3. Construct an angle whose tangent is 1·48.

From B cut off BD = 10 and BC = 14·8.

Then required angle is CDB = 55° (nearly).



16. Given one trigonometrical ratio, to find the others.

Ex. 1. Given $\sin\alpha = \frac{5}{13}$, find $\cos\alpha$ and $\cot\alpha$.

If $\angle BAC = \alpha$,

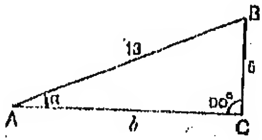
then $13^2 = 5^2 + b^2$,

$\therefore b = \sqrt{13^2 - 5^2}$

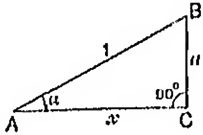
$= 12$,

$\therefore \cos\alpha = \frac{b}{13} = \frac{12}{13}$,

$\therefore \cot\alpha = \frac{b}{5} = \frac{12}{5}$.



Ex. 2. Given that $\cos \alpha = x$, find all the ratios.



$$1^2 = a^2 + x^2,$$

$$\therefore\ a = \sqrt{1 - x^2};$$

$$\therefore\ \sin \alpha = \frac{a}{1} = \sqrt{1 - x^2},$$

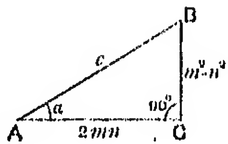
$$\tan \alpha = \frac{a}{x} = \frac{\sqrt{1 - x^2}}{x},$$

$$\operatorname{cosec} \alpha = \frac{1}{a} = \frac{1}{\sqrt{1 - x^2}},$$

$$\cot \alpha = \frac{x}{a} = \frac{x}{\sqrt{1 - x^2}},$$

$$\sec \alpha = \frac{1}{x}.$$

Ex. 3. Given that $\tan \alpha = \dfrac{m^2 - n^2}{2mn}$, find $\sin \alpha$ and $\sec \alpha$.



$$c^2 = (m^2 - n^2)^2 + (2mn)^2$$

$$= m^4 + 2m^2n^2 + n^4,$$

$$\therefore\ c = m^2 + n^2,$$

$$\therefore\ \sin \alpha = \frac{m^2 - n^2}{c} = \frac{m^2 - n^2}{m^2 + n^2},$$

$$\sec \alpha = \frac{c}{2mn} = \frac{m^2 + n^2}{2mn}.$$

EXAMPLES V.

Draw angles of the following magnitude with protractors:

1. Angle 14°. Find the sine.

2. Angle 18°. Find the sine and cosine. Thence show roughly that

$$\sin^2 18° + \cos^2 18° = 1.$$

3. Angle 54°. Find the tangent and cotangent. Thence prove roughly that

$$\cot 54^\circ = \frac{1}{\tan 54^\circ}.$$

4. Angle 20°. Find the secant.

5. Angle 24°. Find the cosecant.

6. Angle 67° 30′. Find the sine and cosine. Then show roughly that

$$\sin^2 67^\circ\, 30' + \cos^2 67^\circ\, 30' = 1.$$

7. Angle 77°. Find the sine, cosine and tangent. Thence prove roughly that

$$\tan 77^\circ = \frac{\sin 77^\circ}{\cos 77^\circ}.$$

8. Angle 37° 30′. Find the secant.

9. Angle 49°. Find the cotangent.

10. Angle 56° 30′. Find the cosecant.

11. Construct an angle whose sine is $\frac{11}{17}$. Measure the angle to the nearest degree.

12. Construct an angle whose sine is $\frac{12}{17}$. Measure the angle to the nearest degree.

13. Construct an angle whose sine is $\frac{13}{17}$. Measure the angle to the nearest degree.

14. From the last three examples, what conclusion do you draw between the variation in the sine of an angle and the variation of the size of the angle?

15. Construct an angle whose cosine is $\frac{3}{5}$. Measure the angle to the nearest degree.

16. Construct an angle whose tangent is ·75. Measure the angle to the nearest degree.

17. Construct an angle whose cosecant is 1·52. Measure the angle to the nearest degree.

18. Construct an angle whose cotangent is 1·48. Measure the angle to the nearest degree.

19. Construct an angle whose secant is 1·78. Measure the angle to the nearest degree.

20. Construct an angle whose sine is $\frac{4}{7}$. Measure the angle to the nearest degree.

---

21. If $\tan\theta = \frac{2}{5}$, find $\sin\theta$ and $\cos\theta$.

22. If $\sin\theta = \frac{3}{10}$, find $\cot\theta$ and $\operatorname{cosec}\theta$.

23. If $\sec\theta = \frac{5}{3}$, find $\cos\theta$ and $\tan\theta$.

24. If $\cos\theta = \frac{3}{8}$, find $\cot\theta$ and $\sin\theta$.

25. If $\cot\theta = \dfrac{a}{b}$, find $\sin\theta$ and $\cos\theta$.

26. Given that $\sin\theta = \frac{1}{2}$, find the value of $1 + \tan^2\theta$.

27. Given that $\sec A = \frac{5}{2}$, evaluate $1 + \cot^2 A$.

28. If $\sin A = s$, express all the other trigonometrical ratios in terms of s.

29. If $\tan A = t$, express all the other trigonometrical ratios in terms of t.

30. If $\sec A = \frac{5}{3}$, find the value of $\dfrac{\cot^2 A - \tan^2 A}{\cot^2 A + \tan^2 A}$.

31. If $(1 + a^2)\cot A = 1 - a^2$, find the values of $\sin A$ and $\sec A$.

32. Given that $\sqrt{mn + m^2}\operatorname{cosec} A = m + n$, find the values of $\tan A$ and $\cos A$.

33. If $\sin A = \frac{8}{17}$, find the value of $\dfrac{\sec A - \tan A}{\sec A + \tan A}$.

34. If $\sec\alpha = \frac{13}{5}$, evaluate $\dfrac{\operatorname{cosec}\alpha - \cot\alpha}{\operatorname{cosec}\alpha + \cot\alpha}$.

35. If $\operatorname{cosec} A = \dfrac{y}{x}$, evaluate $\sec^2 A + \cos^2 A - 1$.

36. If the angles A and B are complementary (*i.e.* $A + B = 90°$) and $\sin A = \dfrac{m}{n}$, find the value of $\sin A\cos B + \cos A\sin B$.

If the angles A and B are complementary and $\cos A = \frac{p}{q}$, value of $\cos A \cos B - \sin A \sin B$.

If $\sin A = \frac{x}{y}$ and $\sin B = \frac{p}{q}$, find the value of

$$\sin A \cos B - \cos A \sin B.$$

If $A + B = 90°$ and $\sin A = \frac{1}{2}$, find the value of

$$\frac{\cot A \cot B - 1}{\cot A + \cot B}.$$

If $\tan A = \sqrt{3}$ and $\tan B = \frac{1}{\sqrt{3}}$, find the value of

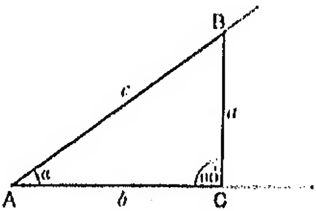
$$\frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

aneous Examples on Chapters I and II start on Test Paper I.

CHAPTER III.

RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS.

17. LET ABC be a right-angled triangle.



Then $\sin\alpha = \dfrac{a}{c}$; $\operatorname{cosec}\alpha = \dfrac{c}{a}$;

$$\therefore \ \sin\alpha = \frac{1}{\operatorname{cosec}\alpha}$$

and $$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha},$$

also $$\cos\alpha = \frac{b}{c}; \quad \sec\alpha = \frac{c}{b};$$

$$\therefore \ \cos\alpha = \frac{1}{\sec\alpha}; \quad \sec\alpha = \frac{1}{\cos\alpha},$$

also $$\tan\alpha = \frac{a}{b}; \quad \cot\alpha = \frac{b}{a},$$

$$\therefore \ \tan\alpha = \frac{1}{\cot\alpha}; \quad \cot\alpha = \frac{1}{\tan\alpha},$$

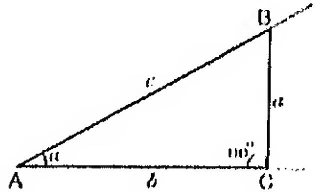
$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \tan\alpha,$$

$$\frac{\cos\alpha}{\sin\alpha} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} = \cot\alpha.$$

18. To prove

$$\sin^2\alpha + \cos^2\alpha = 1$$

where $\sin^2\alpha$ stands for $\{\sin\alpha\}^2$.



$$\sin^2\alpha = \left\{\frac{a}{c}\right\}^2$$

$$\cos^2\alpha = \left\{\frac{b}{c}\right\}^2;$$

$$\therefore \sin^2\alpha + \cos^2\alpha = \frac{a^2+b^2}{c^2} = 1. \quad \text{(By Geometry.)}$$

19. To prove

$$\sec^2\alpha = \tan^2\alpha + 1,$$

and

$$\operatorname{cosec}^2\alpha = \cot^2\alpha + 1.$$

$$\sec^2\alpha = \frac{c^2}{b^2} = \frac{a^2+b^2}{b^2} \text{ (by Geometry)}$$

$$= \frac{a^2}{b^2} + 1 = \tan^2\alpha + 1,$$

$$\operatorname{cosec}^2\alpha = \frac{c^2}{a^2} = \frac{a^2+b^2}{a^2}$$

$$= 1 + \frac{b^2}{a^2} = 1 + \cot^2\alpha.$$

ILLUSTRATIVE EXAMPLES.

Ex. 1. To prove

$$\sin^2 A \operatorname{cosec} A + \cos^2 A \sec A = \sin A + \cos A.$$

$\sin^2 A \operatorname{cosec} A + \cos^2 A \sec A$

$$= \sin^2 A \cdot \frac{1}{\sin A} + \cos^2 A \cdot \frac{1}{\cos A}$$

$$= \sin A + \cos A.$$

Ex. 2. To prove

$$\cot^2 A \tan A \sin A + \tan^2 A \cot A \cos A = \cos A + \sin A.$$

$$\begin{aligned}\cot^2 A \tan A \sin A &+ \tan^2 A \cot A \cos A \\ &= \frac{\cos^2 A}{\sin^2 A} \cdot \frac{\sin A}{\cos A} \cdot \sin A + \frac{\sin^2 A}{\cos^2 A} \cdot \frac{\cos A}{\sin A} \cdot \cos A \\ &= \cos A + \sin A.\end{aligned}$$

Ex. 3. To prove

$$\cos^6 \theta + \sin^6 \theta = 1 - 3 \sin^2 \theta \cos^2 \theta.$$

$$\begin{aligned}\cos^6 \theta + \sin^6 \theta &= (\cos^2 \theta + \sin^2 \theta)(\cos^4 \theta + \sin^4 \theta - \sin^2 \theta \cos^2 \theta) \\ &= (1)\{(\cos^2 \theta + \sin^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta\} \\ &= 1 - 3 \sin^2 \theta \cos^2 \theta.\end{aligned}$$

Ex. 4. To prove

$$(1 - \tan \theta)^2 + (1 - \cot \theta)^2 = (\sec \theta - \operatorname{cosec} \theta)^2.$$

$$\begin{aligned}(1 - \tan \theta)^2 &+ (1 - \cot \theta)^2 \\ &= 1 + \tan^2 \theta - 2 \tan \theta + 1 + \cot^2 \theta - 2 \cot \theta \\ &= \sec^2 \theta - 2 \tan \theta + \operatorname{cosec}^2 \theta - 2 \cot \theta \\ &= \sec^2 \theta + \operatorname{cosec}^2 \theta - 2 \frac{\sin \theta}{\cos \theta} - \frac{2 \cos \theta}{\sin \theta} \\ &= \sec^2 \theta + \operatorname{cosec}^2 \theta - \frac{2(\sin^2 \theta + \cos^2 \theta)}{\sin \theta \cos \theta} \\ &= \sec^2 \theta + \operatorname{cosec}^2 \theta - \frac{2}{\sin \theta \cos \theta} \\ &= (\sec \theta - \operatorname{cosec} \theta)^2.\end{aligned}$$

Ex. 5. To prove

$$\sin^2 \theta + \tan^2 \theta = \sec^2 \theta - \cos^2 \theta.$$

$$\begin{aligned}\sec^2 \theta - \cos^2 \theta &= \tan^2 \theta + 1 - \cos^2 \theta \\ &= \tan^2 \theta + \sin^2 \theta.\end{aligned}$$

Ex. 6. To prove

$$\sin^4 \theta + \sin^2 \theta = 2 - 3 \cos^2 \theta + \cos^4 \theta.$$

$$\begin{aligned}\sin^4 \theta + \sin^2 \theta &= (1 - \cos^2 \theta)^2 + 1 - \cos^2 \theta \\ &= 2 - 3 \cos^2 \theta + \cos^4 \theta.\end{aligned}$$

Ex. 7. Eliminate θ between

$$x = x' \cos\theta - y' \sin\theta,$$

and $$y = x' \sin\theta + y' \cos\theta.$$

Squaring and adding we have

$$x^2 + y^2 = x'^2(\sin^2\theta + \cos^2\theta) + y'^2(\sin^2\theta + \cos^2\theta) - 2x'y' \sin\theta\cos\theta$$
$$+ 2x'y' \sin\theta\cos\theta$$
$$= x'^2 \qquad\qquad + y'^2.$$

The equation $x^2 + y^2 = x'^2 + y'^2$ is called the *Eliminant.*

EXAMPLES VI.

Prove that

1. $\sin A \tan A = \dfrac{1-\cos^2 A}{\cos A}$.

2. $\tan A \cot A \sec A = \dfrac{1}{\cos A}$.

3. $(\sin\alpha + \cos\alpha)^2 = 1 + 2\sin\alpha\cos\alpha$.

4. $(\sin\alpha - \cos\alpha)^2 = 1 - 2\sin\alpha\cos\alpha$.

5. $\dfrac{1}{\tan^2\theta} + 1 = \operatorname{cosec}^2\theta$.

6. $\dfrac{1}{\cot^2\theta} + 1 = \dfrac{1}{\cos^2\theta}$.

7. $1 - 4\sin^2\theta = 4\cos^2\theta - 3$.

8. $1 + 3\tan^2 A = \dfrac{1 + 2\sin^2 A}{\cos^2 A}$.

9. $3 + 4\cot^2 A = 3\operatorname{cosec}^2 A + \dfrac{\cos^2 A}{\sin^2 A}$.

10. $\tan^2 A - \sin^2 A = \dfrac{\sin^4 A}{\cos^2 A}$.

11. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.

12. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$.

13. $\dfrac{1}{\sec A + \tan A} = \dfrac{\cos A}{1 + \sin A}$.

14. $\dfrac{\cos^2 A}{1 + \sin A} = 1 - \sin A$.

15. $1 - \cos^4 A = \sin^2 A\,(1 + \cos^2 A)$.

16. $\dfrac{\cot^2 \alpha}{\cot^2 \alpha + 1} = \cos^2 \alpha$.

17. $\tan \alpha \sin \alpha + \cos \alpha = \sec \alpha$.

18. $\sec^4 \alpha - \tan^4 \alpha = \sec^2 \alpha + \tan^2 \alpha$.

19. $\dfrac{\sin^2 A}{\cos A} + \dfrac{\tan A}{\cot A} = \dfrac{\sin^2 A\,(1 + \cos A)}{\cos^2 A}$.

20. $\cot^2 \alpha + \tan^2 \alpha - \sin^2 \alpha = \dfrac{\cos^4 \alpha + \sin^6 \alpha}{\cos^2 \alpha \sin^2 \alpha}$.

21. $(\cot A + \operatorname{cosec} A)^2 = \dfrac{1 + \cos A}{1 - \cos A}$.

22. $\cos^2 \theta \operatorname{cosec}^2 \theta = \operatorname{cosec}^2 \theta - 1$.

23. $(\sec^2 A - 1) \cos^2 A = \sin^2 A$.

24. $(1 + \cot^2 A)(1 + \tan^2 A) \sin^2 A = \dfrac{1}{\cos^2 A}$.

25. $\sqrt{1 - \sin^2 \theta} \tan \theta = \sin \theta$.

26. $\sin^3 \alpha - \cos^3 \alpha = (\sin \alpha - \cos \alpha)(1 + \sin \alpha \cos \alpha)$.

27. $\dfrac{\sin \theta}{1 + \cos \theta} + \dfrac{1 + \cos \theta}{\sin \theta} - \dfrac{2}{\sin \theta} = 0$.

28. $\dfrac{\tan^2 \theta - \cot^2 \theta}{1 + \cot^2 \theta} = \dfrac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta}$.

29. $\dfrac{1}{1 + \tan A} = \dfrac{\cot A}{1 + \cot A}$.

30. $1 - \sin^2 \alpha = \operatorname{covers} \alpha\,(1 + \sin \alpha)$.

31. $(1 + \cos \alpha)(1 - \cos \alpha)^2 = \sin^2 \alpha \operatorname{vers} \alpha$.

32. $\dfrac{1}{1+\cos\alpha}+\dfrac{1}{1-\cos\alpha}=2\operatorname{cosec}^2\alpha.$

33. $\operatorname{cosec}^4\alpha-\operatorname{cosec}^2\alpha=\cot^4\alpha+\cot^2\alpha.$

34. $\sec^4\alpha-\sec^2\alpha=\dfrac{\sin^2\alpha}{\cos^4\alpha}.$

35. $(\sin A-\cos A)(\tan A+\cot A)=\sec A-\operatorname{cosec} A.$

36. $(\sec A-\operatorname{cosec} A)(\tan A+\cot A)$

$$=(\sin A-\cos A)\sec^2 A\operatorname{cosec}^2 A.$$

37. $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta.$

38. $\dfrac{\cot\theta+\operatorname{cosec}\theta}{\operatorname{cosec}\theta-\cot\theta}=\dfrac{\sin^2\theta}{(1-\cos\theta)^2}.$

39. $\operatorname{vers} A(1+\sec A)=\sin^2 A\sec A.$

40. $(\cot A-\tan A)\sin A=2\cos A-\sec A.$

41. $\dfrac{\tan^2 A+\cot^2 A}{\tan^2 A-\cot^2 A}=\dfrac{\sin^4 A+\cos^4 A}{\sin^2 A-\cos^2 A}.$

42. $\dfrac{2\tan A}{1+\tan^2 A}=2\sin A\cos A.$

43. $\dfrac{1-\tan^2 A}{1+\tan^2 A}=2\cos^2 A-1.$

44. $(\sec^2 A+\tan^2 A)(\operatorname{cosec}^2 A+\cot^2 A)=1+2\sec^2 A\operatorname{cosec}^2 A.$

45. $\dfrac{\sec A+\tan A}{\operatorname{cosec} A+\cot A}-\dfrac{\sec A-\tan A}{\operatorname{cosec} A-\cot A}=2(\sec A-\operatorname{cosec} A).$

46. $\dfrac{\sin^3 A+\cos^3 A}{\sin A+\cos A}+\dfrac{\sin^3 A-\cos^3 A}{\sin A-\cos A}=2.$

47. $(1+\sec\theta+\tan\theta)(1+\operatorname{cosec}\theta+\cot\theta)$

$$=2(1+\tan\theta+\cot\theta+\sec\theta+\operatorname{cosec}\theta).$$

48. $(\operatorname{cosec} A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}.$

49. $\dfrac{\tan\theta}{(1+\tan^2\theta)^2}+\dfrac{\cot\theta}{(1+\cot^2\theta)^2}=\sin\theta\cos\theta.$

50. $\cos^4\theta+\cos^2\theta=2-3\sin^2\theta+\sin^4\theta.$

Eliminate θ from the following:

51. $x=r\cos\theta;\quad y=r\sin\theta.$

52. $x=a\cos\theta;\quad y=b\sin\theta.$

53. $x=a\sec\theta;\quad y=b\tan\theta.$

54. $x=a\operatorname{cosec}\theta;\quad y=b\cot\theta.$

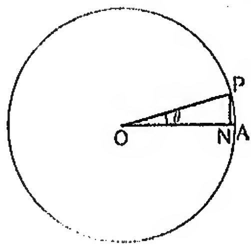
55. $x\sec\theta=x'-y'\tan\theta,$

$y\sec\theta=x'\tan\theta+y'.$

CHAPTER IV.

TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES.

20. 0°.



Let $A\hat{O}P$ be a small angle; PN the perpendicular to OA.

Let the radius OP or OA be the unit of length.

Then NP measures $\sin\theta$,

ON „ $\cos\theta$.

When OP moves towards OA and ultimately coincides with it, θ becomes $0°$; NP becomes 0; ON becomes 1 = the unit of length.

$\therefore \sin 0° = 0.$

$\therefore \cos 0° = 1.$ $\qquad\qquad \therefore \sec 0° = 1.$

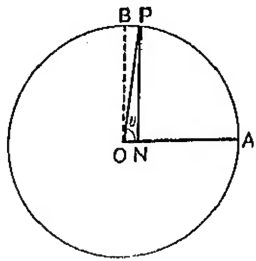
$$\therefore \tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0.$$

Cot θ and cosec θ increase without limit as θ approaches 0 and by taking θ sufficiently near zero, cot θ and cosec θ may be made as great as desired: this is expressed shortly

$$\cot\theta = \text{infinity} = \infty,$$

$$\operatorname{cosec}\theta = \text{infinity} = \infty.$$

21. 90°.



Let $A\hat{O}P$ be an angle nearly $90°$; PN the perpendicular to OA.

Let the radius OP or OA be the unit of length.

Then NP measures $\sin\theta$,

ON " $\cos\theta$.

Draw OB perpendicular to OA.

When P moves towards B and coincides with B

$$\theta = 90°; \quad NP = OB = 1; \quad ON = 0.$$

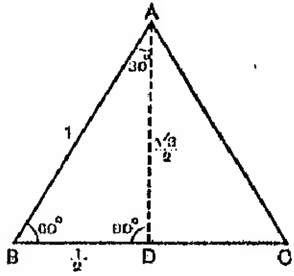
$\therefore \sin 90° = 1.$ $\qquad \therefore \operatorname{cosec} 90° = 1.$

$\cos 90° = 0.$ $\qquad \sec 90° = \infty.$

$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} = \infty. \qquad \cot 90° = 0.$$

22. 30° and 60°.

Let ABC be an equilateral triangle whose sides are the unit of length.



Draw AD perpendicular to BC.

Then by Geometry $BD = \frac{1}{2}$;

$$AD = \sqrt{AB^2 - BD^2} = \frac{\sqrt{3}}{2}$$

$$A\hat{B}D = 60^\circ \text{ and } B\hat{A}D = 30^\circ.$$

$$\sin 60^\circ = \frac{DA}{BA} = \frac{\sqrt{3}}{2} \qquad \therefore \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}$$

$$\cos 60^\circ = \frac{BD}{BA} = \frac{1}{2} \qquad \therefore \sec 60^\circ = 2$$

$$\tan 60^\circ = \frac{DA}{BD} = \sqrt{3} \qquad \therefore \cot 60^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 30^\circ = \frac{BD}{AB} = \frac{1}{2} \qquad \therefore \operatorname{cosec} 30^\circ = 2$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2} \qquad \therefore \sec 30^\circ = \frac{2}{\sqrt{3}}$$

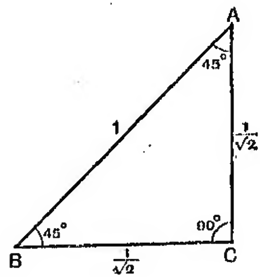
$$\tan 30^\circ = \frac{BD}{AD} = \frac{1}{\sqrt{3}} \qquad \therefore \cot 30 = \sqrt{3}.$$

The student should note that

$$\sin 60^\circ = \cos 30^\circ$$
$$\cos 60^\circ = \sin 30^\circ$$
$$\tan 60^\circ = \cot 30^\circ.$$

He will learn later that these properties always hold for complementary angles, *i.e.* angles whose sum = 90°.

23. 45°.



Let ABC be a right-angled triangle having BC = AC and AB the unit of length.

Then by Geometry

$$AC^2 + BC^2 = AB^2 = 1,$$

$$\therefore\ 2AC^2 = 1,$$

$$\therefore\ BC = AC = \frac{1}{\sqrt{2}};$$

then $\sin 45° = \dfrac{AC}{AB} = \dfrac{1}{\sqrt{2}}$ $\quad\therefore\ \operatorname{cosec} 45° = \sqrt{2}$

$\cos 45° = \dfrac{BC}{AB} = \dfrac{1}{\sqrt{2}}.$ $\quad\therefore\ \sec 45° = \sqrt{2}$

$\tan 45° = \dfrac{AC}{BC} = 1$ $\quad\therefore\ \cot 45° = 1.$

24. It is useful to know these values and the best way to do so is to recall mentally the figure. For reference a table is given.

✓

	$0°$	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

25. In actual practice **Tables** of Natural Sines, cosines etc. are always employed.

In 4-figure tables the sines, cosines etc. are given for all angles between 0° and 90° at intervals of 6 minutes, difference columns being provided for angles of 1, 2, 3, 4, 5 minutes.

It is proved in Art. 42 etc. that the sine, tangent and secant increase as the angle increases from 0° to 90°, while the cosine, cotangent and cosecant diminish as the angle increases from 0° to 90°; thus the numbers found in the difference columns are *added* in the case of the sine, tangent and secant and *subtracted* in the case of the cosine, cotangent and cosecant.

Ex. 1. Find the value of sin 17° 38′.

Turning to the page of Natural Sines, we look in the first column for 17° and along the row containing 17° to the number in the column headed by 30′ (the number of minutes next below that required). We now have to find the difference for 2′, and looking in the difference column headed by 2 and in the same row as before we see the number 6.

$$\therefore \sin 17° 30' = \cdot 3024$$

difference for 2′ $= \cdot 0006.$

$\therefore$ *Adding* $\sin 17^\circ\,38' = \cdot 3030.$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14

Ex. 2. Find the value of $\cos 58^\circ\,28'$.

As in Ex. 1, we find that

$$\cos 58^\circ\,24' = \cdot 5240$$

and the difference for $4'$ $= \cdot 0010.$

$\therefore$ *Subtracting* $\cos 58^\circ\,28' = \cdot 5230.$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
58	5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12

Ex. 3. Find x, given that $\tan x = \cdot 5275$.

On turning to the page of Natural Tangents and selecting the number nearest to ·5275 and *smaller* than it, we find that

$$\tan 27^\circ\,48' = \cdot 5272.$$

There is now a difference of 3 in the last figure to be accounted for, and looking for the number nearest to 3 in the Difference Columns we find 4, which is in the column headed by $1'$;

$$\therefore\ x = 27^\circ\,49'.$$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18

Ex. 4. Find x, given that $\cot x = 1{\cdot}6211$.

From the table of Natural Cotangents, selecting the number nearest to 1·6211 and *smaller* than it, we find that

$$\cot 31^\circ\, 42' = 1{\cdot}6191.$$

The difference to be accounted for is 20, and in the difference columns the nearest number to this is 21, in the column headed by $2'$.

Since as the value of the cotangent increases the angle gets smaller, we *subtract*;

$$\therefore\ \cot 31^\circ\, 40' = 1{\cdot}6211.$$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1 2 3	4 5
31	1·6643	6577	6512	6447	6383	6319	6255	6191	6128	6066	11 21 32	43 53

26. We shall see in Art. 27 that tables of sines and tangents from 0° to 90° would really be sufficient, for

$$\cos \mathsf{A} = \sin (90^\circ - \mathsf{A})$$

$$\sec \mathsf{A} = \frac{1}{\cos \mathsf{A}} = \frac{1}{\sin (90^\circ - \mathsf{A})}$$

$$\cot \mathsf{A} = \frac{1}{\tan \mathsf{A}}$$

$$\operatorname{cosec} \mathsf{A} = \frac{1}{\sin \mathsf{A}}.$$

Thus all the other four ratios can be reduced to expressions involving only sines and tangents.

EXAMPLES VII.

Prove that

1. $\sin 60° \cos 30° + \sin 30° \cos 60° = \sin 90°$.
2. $\sin 60° \cos 30° - \sin 30° \cos 60° = \sin 30°$.
3. $\dfrac{\tan 60° + \tan 30°}{1 - \tan 60° \tan 30°} = \tan 90°$.
4. $\dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°} = \tan 30°$.
5. $2\cos^2 30° - 1 = 1 - 2\sin^2 30° = \cos 60°$.
6. $2\sin 30° \cos 30° = \sin 60°$.
7. $3\sin 30° - 4\sin^3 30° = \sin 90°$.
8. $4\cos^3 30° - 3\cos 30° = \cos 90°$.
9. $\cos^2 45° - \sin^2 45° = \cos 90°$.
10. $\operatorname{cosec} 60° \cot 30° \tan 60° = 2\sec^2 45° \cos 30°$.
11. $\frac{1}{4}\sec^2 30° + 3\cos^2 45° = \sec 60° - \frac{1}{2}\tan^2 30°$.
12. $\cos^2 \dfrac{\pi}{6} - \sin^2 \dfrac{\pi}{6} = \cos \dfrac{\pi}{3}$.
13. $\cos \dfrac{\pi}{4} \sin \dfrac{\pi}{4} - \sin^2 \dfrac{\pi}{6} = \cos^2 \dfrac{\pi}{3}$.
14. $\cot^2 \dfrac{\pi}{6} - \tan^2 \dfrac{\pi}{6} = \dfrac{\sin^2 \dfrac{\pi}{3} - \cos^2 \dfrac{\pi}{3}}{\cos^2 \dfrac{\pi}{3} \cos^2 \dfrac{\pi}{6}}$.

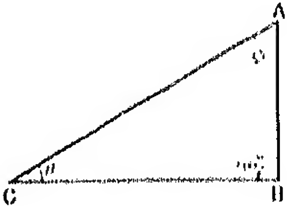
Find the value of

15. $\sin^2 30° + \sin^2 60° + \tan^2 45°$.
16. $2\cos 30° \cos 45° \sin 30° \sin 45°$.
17. $\tan^2 30° + 4\sin^2 45° + \frac{1}{3}\cos^2 30°$.
18. $\cos^2 \dfrac{\pi}{3} + \sin^2 \dfrac{\pi}{4} + \sin^2 \dfrac{\pi}{2}$.
19. $2\sin \dfrac{\pi}{4} + \dfrac{1}{2} \operatorname{cosec} \dfrac{\pi}{4}$.
20. $\tan^2 \dfrac{\pi}{3} + 4\cos^2 \dfrac{\pi}{4} + 3\operatorname{cosec}^2 \dfrac{\pi}{3}$.

27. Complementary angles.

Two angles are said to be complementary when their sum is a right angle; and either angle is said to be the complement of the other.

Thus in every right-angled triangle two of the angles must be complementary; and any two complementary angles can be drawn so that they form two angles of a right-angled triangle.



Let ABC be any right-angled triangle, then θ and ϕ are complementary.

$$\sin\theta = \frac{AB}{AC} = \cos\phi, \quad \therefore \quad \operatorname{cosec}\theta = \sec\phi,$$

$$\cos\theta = \frac{CB}{CA} = \sin\phi, \quad \therefore \quad \sec\theta = \operatorname{cosec}\phi,$$

$$\tan\theta = \frac{AB}{BC} = \cot\phi.$$

These relations may be put into words thus:

The *sine* of an angle = the *cosine* of its *complement*.

The *cosine* of an angle = the *sine* of its *complement*, etc.

Notice $\qquad \phi = 90° - \theta.$

$$\therefore \quad \sin\theta = \cos(90° - \theta); \quad \cos\theta = \sin(90° - \theta);$$

$$\tan\theta = \cot(90° - \theta), \text{ etc.}$$

EXAMPLES VIII.

1. Given $\sin 17° = \cdot 2924$
and $\sin 73° = \cdot 9563$
find $\tan 17°$ and $\tan 73°$.

2. Given $\cos 22° = \cdot 9272$
and $\cos 68° = \cdot 3746$
find $\tan 22°$ and $\tan 68°$.

3. Given $\tan 35° = \cdot 7002$
and $\cos 35° = \cdot 8192$
find $\cos 55°$ and $\tan 55°$.

Prove the following identities:

4. $\cos(90^\circ - A)\tan(90^\circ - A) = \cos A.$

5. $\tan\theta + \cot\theta = \operatorname{cosec}(90^\circ - \theta)\operatorname{cosec}\theta.$

6. $\sin(90^\circ - \theta)\cot(90^\circ - \theta)\cot\theta\sec\theta = 1.$

7. $\dfrac{\sin 62^\circ}{\sec 62^\circ}\cdot\dfrac{\cot 28^\circ}{\cos 28^\circ} = \cos 28^\circ.$

8. $\dfrac{\tan^2 43^\circ . \sin^2 43^\circ}{\cot 47^\circ + \cos 47^\circ} = \tan 43^\circ - \cos 47^\circ.$

9. $\sin^2 A \operatorname{cosec}\left(\dfrac{\pi}{2} - A\right) - \cot^3\left(\dfrac{\pi}{2} - A\right)\cos A = 0.$

10. $\cot\left(\dfrac{\pi}{2} - \theta\right) + \cot\theta = \tan\theta\operatorname{cosec}^2\theta.$

11. If $A = 30^\circ$, prove that

 (i) $\cos 2A = 2\cos^2 A - 1,$

 (ii) $\cos 3A = 4\cos^3 A - 3\cos A,$

 (iii) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}.$

12. If $A = 30^\circ$ and $B = 60^\circ$, prove that

 (i) $\sin(A + B) = \sin A\cos B + \cos A\sin B,$

 (ii) $\cos(B - A) = \cos A\cos B + \sin A\sin B.$

Prove that

13. $\tan(90^\circ - A)\sin(90^\circ - A) = \cos^2 A\operatorname{cosec} A.$

14. $\tan A\sec(90^\circ - A) - \sin^2 A\operatorname{cosec}(90^\circ - A) = \cos A.$

15. $\operatorname{cosec}(90^\circ - A) - \tan A\sin(90^\circ - A)\cot(90^\circ - A) = \cos A.$

16. $\dfrac{\sin 68^\circ}{\sec 68^\circ}\cdot\dfrac{\tan 68^\circ}{\cos 22^\circ} = \cos 22^\circ.$

17. $\dfrac{\operatorname{cosec}^2 33^\circ\tan^2 33^\circ}{\cot 57^\circ}\cdot\dfrac{\cot 33^\circ}{\sec^2 33^\circ} = \sec^2 57^\circ - 1.$

18. $\cot 67^\circ\cot 23^\circ\cos 67^\circ\tan 67^\circ = \cos 23^\circ.$

19. $\cos 13^\circ\tan 13^\circ\tan 77^\circ\operatorname{cosec} 77^\circ = 1.$

20. $\sin(90^\circ - A)\cos(90^\circ - A)\cot A - 1 + \cos^2(90^\circ - A) = 0.$

28. Trigonometrical equations.

Ex. 1. Solve $$3\,\text{cosec}\,\theta = 4\sin\theta.$$

$$\therefore\ 3\cdot\frac{1}{\sin\theta} = 4\sin\theta.$$

$$\therefore\ \sin^2\theta = \tfrac{3}{4}.$$

$$\therefore\ \sin\theta = \pm\frac{\sqrt{3}}{2}.$$

Now $$\sin 60° = \frac{\sqrt{3}}{2}.$$

$\therefore$ one solution of the equation is 60°.

N.B. The student will learn later that there are further solutions.

Ex. 2. Solve

$$\sec^2\theta + 5 - 3\sqrt{3}\tan\theta = 0.$$

Now $$\sec^2\theta = 1 + \tan^2\theta.$$

$$\therefore\ \tan^2\theta + 6 - 3\sqrt{3}\tan\theta = 0.$$

$$\therefore\ (\tan\theta - \sqrt{3})(\tan\theta - 2\sqrt{3}) = 0.$$

$\therefore$ either $$\tan\theta = \sqrt{3},\ \textit{i.e.}\ \theta = 60°,$$

or $$\tan\theta = 2\sqrt{3} = 3{\cdot}4641.$$

Now we find from the Tables

$$\tan 73°\,54' = 3{\cdot}4646.$$

$$\therefore\ \theta = 73°\,54'\ \text{(approx.)}.$$

Ans. 60° or 73° 54′.

Ex. 3. Solve

$$3\tan^2\theta - 8\tan\theta\sec\theta + 16\tan\theta - 6\sec\theta + 3 = 0.$$

Then $$\frac{3\sin^2\theta}{\cos^2\theta} - \frac{8\sin\theta}{\cos^2\theta} + \frac{16\sin\theta}{\cos\theta} - \frac{6}{\cos\theta} + 3 = 0.$$

$$\therefore\ 3\sin^2\theta - 8\sin\theta + 16\sin\theta\cos\theta - 6\cos\theta + 3\cos^2\theta = 0.$$

$$\therefore\ 3(\sin^2\theta + \cos^2\theta) - 8\sin\theta - 6\cos\theta + 16\sin\theta\cos\theta = 0.$$

$$\therefore\ (3 - 8\sin\theta)(1 - 2\cos\theta) = 0.$$

$\therefore$ either $\cos\theta = \frac{1}{2}$, *i.e.* $\theta = 60°$,

or $\sin\theta = \frac{3}{8} = \cdot 375$;

from Tables $\sin 22° 1' = \cdot 375$ (approx.),

$\therefore \theta = 22° 1'$.

Ans. 60° or 22° 1′.

EXAMPLES IX.

Solve the following equations, *i.e.* find values from Table Art. 24 or if necessary from Tables of Natural Sines, etc., which satisfy them.

1. $3\sec\theta = 4\cos\theta$.
2. $2\sin\theta = \operatorname{cosec}\theta$.
3. $3\cot\theta = \tan\theta$.
4. $\sqrt{3}\sec\theta = 2\tan\theta$.
5. $\operatorname{cosec}\theta = 2\cot\theta$.
6. $\sec\theta + 2\tan\theta = 2\sin^2\theta\sec\theta + 2\cos\theta$.
7. $\sin^2\theta + 2\sin\theta = 2 - \cos^2\theta$.
8. $7\cos^2\theta - 17\cos\theta + 22 = 16 - \sin^2\theta$.
9. $\operatorname{cosec}^2\theta + 5 - 3\sqrt{3}\cot\theta = 0$.
10. $\sec^2\theta + 1 - 3\tan\theta = 0$.
11. $3\tan^2\theta = 1 + \sec^2\theta$.
12. $\sec^2\theta + \tan^2\theta = \frac{5}{3}$.
13. $\operatorname{cosec}^2\theta + \cot^2\theta = \frac{5}{3}$.
14. $\tan\theta + 3\cot\theta = 5\sec\theta$.

15. $15 \sin \theta + 2 \cos^2 \theta - 9 = 0.$

16. $9 (\cos^2 \theta + \sin \theta) = 11.$

17. $\tan^2 \theta (3 - 2 \sec \theta) = 3 (\sec \theta - 1).$

18. $\dfrac{4 - \sin \theta}{1 - \sin \theta} - \dfrac{25}{4} \sec^2 \theta + \dfrac{2}{1 + \sin \theta} = 0.$

19. $9 \sin^2 \theta + 27 \sin \theta = 10.$

20. $\cos^2 \theta - 2 \sin \theta + \frac{1}{4} = 0.$

21. $\dfrac{\sin \theta}{1 + \cos \theta} = 2 - \cot \theta.$

22. $16 \sin^4 \theta - 16 \sin^2 \theta + 3 = 0.$

23. $\tan \theta + \cot \theta = \dfrac{4}{\sqrt{3}}.$

24. $2 \sin^2 \theta + 3 \sin \theta - 4 = 0.$

25. $3 \sin^2 \theta - 4 \sin \theta + 1 = 0.$

Miscellaneous Examples on Chapters III and IV start on page 206, Test Paper IX.

CHAPTER V.

EASY PROBLEMS.

29. In these problems the terms

"Angle of **Elevation**,"

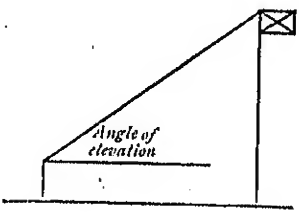
"Angle of **Depression**"

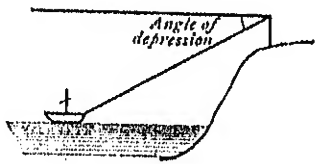
are often used.

Def. The angle between a horizontal plane through an observer's eye and a line joining the eye to any object is called

(i) The angle of Elevation of the object when it is higher than the eye.

(ii) The angle of Depression of the object when it is lower than the eye.





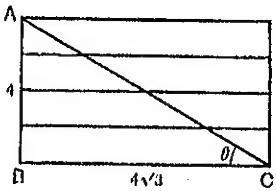
Ex. 1. A five-barred gate is 4 ft. high and $4\sqrt{3}$ ft. long. What is the length of the cross piece and what is its inclination?

From fig. $$\tan\theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}},$$

but $$\tan 30° = \frac{1}{\sqrt{3}}.$$

$$\therefore \theta = 30°.$$

$$\therefore B\hat{A}C = 90° - 30° = 60°.$$



Again $$\frac{AC}{AB} = \operatorname{cosec}\theta.$$

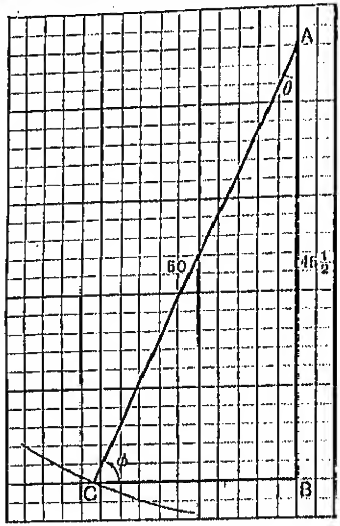
$$\therefore \frac{AC}{4} = \operatorname{cosec} 30° = 2.$$

$$\therefore AC = 8 \text{ ft.}$$

Ans. 8 ft.; 30° to the horizon or 60° to the vertical.

N.B. In this question we have "solved" a right-angled triangle, given the two sides containing the right angle.

Ex. 2. A man wishes to climb a wall $45\frac{1}{2}$ ft. high with a ladder 50 ft. long, find the distance of the foot of the ladder from the foot of the wall and the inclination of the ladder.



Given $\cos 24^\circ 30' = \cdot 91$,

and $\sin 24^\circ 30' = \cdot 4147$.

Let AB be the wall, AC the ladder.

$$\cos \theta = \frac{AB}{AC} = \frac{45\frac{1}{2}}{50}$$

$$= \cdot 91.$$

$$\therefore \quad \theta = 24^\circ 30'.$$

$$\therefore \quad \phi = 65^\circ 30'.$$

$$\frac{CB}{AC} = \sin \theta = \sin 24^\circ 30'$$

$$= \cdot 4147.$$

$$\therefore \quad CB = 50 \times (\cdot 4147)$$

$$= 20 \cdot 735 \text{ ft.}$$

Ans. 20·735 ft.; 65° 30′ to the horizontal or 24° 30′ to the vertical.

N.B. In this question we have solved a right-angled triangle, given one side and the hypotenuse.

Rough check. It is advisable to check the results by a diagram drawn to scale.

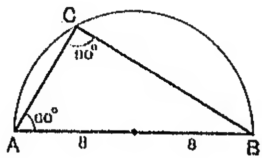
Take AB = 4·55 units on squared paper, with centre A, describe a circle radius 5 units.

By measurement

$$BC = 2{\cdot}1 \text{ units} = 21 \text{ ft.}$$

By protractor $\theta = 25°$.

Ex. 3. A man at sea observes 3 light-ships A, B and C on the horizon, A is directly in front of him, B is directly behind him, and $C\hat{A}B$ is known to be 60°. Find the distances of C from A and B and the angle CBA, assuming the horizon to be a circle 8 miles radius.



$A\hat{C}B = 90°$ being the angle in a semi-circle.

$C\hat{B}A =$ complement of $C\hat{A}B = 30°$.

$$\frac{CA}{AB} = \cos 60° = \frac{1}{2}.$$

$$\therefore\ CA = \tfrac{1}{2}\,.\,AB = 8 \text{ miles.}$$

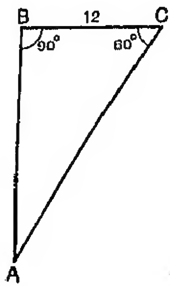
$$\frac{CB}{AB} = \sin 60° = \frac{\sqrt{3}}{2}.$$

$$\therefore\ CB = 8\sqrt{3} \text{ miles.}$$

Ans. 8 miles; $8\sqrt{3}$ (= 13·86) miles; 30°.

N.B. In this question we have solved a right-angled triangle, given the hypotenuse and one angle.

Ex. 4. In a jib-crane, the jib is inclined at 60° to the horizon and the tie-rod, 12 ft. long, is horizontal. Find the length of the jib and the height of the crane.



$$\frac{AC}{BC} = \sec 60°.$$

$$\therefore \quad AC = 12 \sec 60° = 24 \text{ ft.}$$

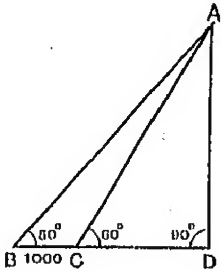
$$\frac{AB}{BC} = \tan 60°.$$

$$\therefore \quad AB = 12 \tan 60° = 12\sqrt{3} \text{ ft.}$$

Ans. 24 ft. and $12\sqrt{3}$ ($= 20{\cdot}78$) ft.

N.B. In this question we have solved a right-angled triangle, given one angle and one side.

Ex. 5. A man observes the elevation of the top of a mountain to be 50°, he walks 1000 feet nearer and finds the elevation to be 60°. Find the height of the mountain to the nearest foot.



$$CD = AD \cot 60° = AD\,(\cdot 5774),$$

$$BD = AD \cot 50° = AD\,(\cdot 8391).$$

$$\therefore\quad BC = BD - CD = AD\,(\cdot 8391 - \cdot 5774)$$

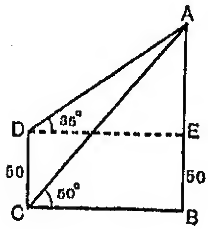
$$= AD\,(\cdot 2617).$$

$$\therefore\quad AD = \frac{1000}{\cdot 2617} = 3821 \text{ ft.}$$

Ans. 3821 ft.

[Examples 5, 6, 7 can be solved by the *Link Method* given in Chap. X.]

Ex. 6. From the deck of a ship the elevation of the top of a cliff is 50°; from the top of a mast 50 ft. high the elevation is 35°. Find the height of the cliff to the nearest foot.



$$CB = AB \cot 50°,$$

$$CB = DE = AE \cot 35° = (AB - 50) \cot 35°,$$

$$\therefore\ AB \cot 50° = (AB - 50) \cot 35°,$$

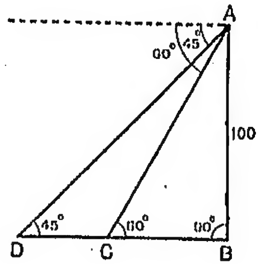
$$\therefore\ AB (\cot 35° - \cot 50°) = 50 \cot 35°,$$

$$\therefore\ AB (1{\cdot}4281 - {\cdot}8391) = 50 (1{\cdot}4281).$$

$$\therefore\ AB = \frac{71{\cdot}405}{{\cdot}589} = 121 \text{ ft.}$$

Ans. 121 ft.

Ex. 7. From the top of a tower 100 ft. high the angles of depression of two objects due north of the tower are 60° and 45°. Find the distance between the objects to the nearest foot.



$$CB = 100 \cot 60° = \frac{100}{\sqrt{3}},$$

$$DB = 100 \cot 45° = 100,$$

$$\therefore \; CD = 100 - \frac{100}{\sqrt{3}} = 42 \text{ ft.}$$

Ans. 42 ft.

EXAMPLES X.

Wherever possible the question should be checked by a diagram drawn to scale. Squared paper will be found useful.

1. A ladder 20 ft. long is placed against a wall so that the foot of the ladder is 10 ft. from the wall. Find the inclination of the ladder to the vertical.

2. A boat is to be launched down a slope whose inclination to the horizon is 30°. The length of the slope is 50 yds., find the height of the boat above the level of the water.

3. A flag-staff 60 ft. high is held up by ropes, each being attached to the top of the flag-staff and to a peg in the ground and inclined at 30° to the vertical; find the lengths of the ropes and the distances of the pegs from the foot of the flag-staff.

4. A kite is flying with the string inclined at 45° to the horizon, find the height of the kite above the ground when the string is 50 yds. long.

5. A man lives in a road inclined at 30° to a river and at a distance of half a mile as the crow flies from the river; how far must he walk along the road to reach the river?

6. The elevation of a tower 600 ft. away is 30°. Find its height to the nearest foot.

7. How far must a man be from a house 40 ft. high in order that it may subtend an angle 60°?

8. A tower casts a shadow 300 ft. long when the sun's altitude is 30°. Find the height of the tower.

9. From the top of a mast of a ship 50 ft. high the angle of depression of an object is 20°, find the distance of the object from the ship ($\cot 20° = 2{\cdot}7475$).

10. A man wishes to find the width of a river, he stands at B immediately opposite an object A on the other side, he then walks 100 yds. along the bank to C and finds the angle BOA to be 40°. What is the width of the river? $\tan 40° = {\cdot}8391$.

11. A man fires a gun at an elevation 3° and hits the top of a target 20 ft. high and 1000 yds. away. How much higher should the target have been if gravity had not acted? Given $\tan 3° = {\cdot}0524$.

12. A tower has an elevation 60° from a point due north of it and 45° from a point due south. If the two points are 100 yds. apart, find the height of the tower and its distance from each point of observation.

13. From the top of a mast 60 ft. high, two buoys are observed due north at angles of depression 50° and 40°. Find the distance between the buoys.

Given $\cot 50° = \cdot 8391$; $\cot 40° = 1 \cdot 1918$.

14. A man observes the elevation of a balloon to be 30°, he then walks one mile towards the balloon and finds the elevation to be 60°; how much further must he walk to be directly underneath the balloon?

15. An observer at a point A sees two forts B and C; he finds $B\hat{A}C = 45°$ and knows that $C\hat{B}A = 90°$. He then walks 2 miles towards B and finds the forts now subtend an angle 60°. How far are the forts apart?

16. In a jib-crane, the jib is inclined at 60° to the horizon and the tie-rod at 30°. If the jib is 40 ft. find the height of the crane and the length of the tie-rod.

17. From the ground the elevation of a cliff is 60°; a man goes up from that point in a captive balloon 100 ft. and finds the elevation to be 50°. Find the height of the cliff to nearest foot. $\cot 50° = \cdot 8391$.

18. From a boat 1000 ft. at sea the elevation of a cliff is 30° and of the top of a building on the edge of the cliff 33°. Find the height of the building and the height of the cliff to nearest foot. $\tan 33° = \cdot 6494$.

19. A man on one bank of a river observes a point on the opposite bank and finds the straight line between himself and the point makes an angle 60° with the current, after walking along the bank 100 ft. the angle is 45°. Find the width of the river to the nearest foot.

In the following examples Tables must be used.

20. A flag-staff has an elevation 50° from a point one side and 40° from a point on the other side directly opposite to the first point. If the two points are 150 feet apart, find the height of the flag-staff to the nearest foot.

21. From a balloon the angles of depression of two buildings due south of the balloon and known to be 1 kilometre apart are 20° and 35°, find the height of the balloon to the nearest metre.

22. An observer at a point A sees two forts B and C; he finds $C\hat{A}B = 50°$ and knows that $B\hat{C}A = 90°$. He then walks half a mile towards C and finds $C\hat{A}B = 55°$. Find the distance between the forts to the nearest tenth of a mile.

23. From a ship 2 kilometres at sea the elevation of a cliff is 20° and of the top of a building on the edge of the cliff 21°. Find the height of the building to the nearest decimetre.

24. A man on one bank of a river observes a point on the opposite bank and finds the straight line between himself and the point makes an angle 14° with the current, after walking 300 feet down the bank the angle is 26°. Find the width of the river to nearest foot.

25. With a vertical stick 12 inches in length, the sun casts a shadow 7 inches long. What is the elevation of the sun?

26. A flag-staff on one bank of a river is viewed by a man immediately opposite on the other bank, and the elevation of the summit is found to be 57°; on retiring 100 feet the elevation becomes 35°. Find the breadth of the river to the nearest foot.

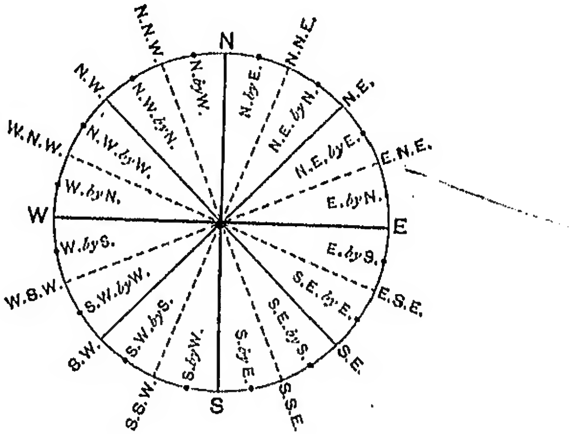
27. ✓ From the foot of a post 12 ft. high, the elevation of the top of a flag-staff is 61°, while from the top it is 52°. Find the height of the flag-staff to the nearest foot.

28. The shadow of a tower is 55 ft. longer when the sun's elevation is 28° than it is when the elevation is 42°. Calculate the length of the shorter shadow to the nearest foot.

29. The line joining two buildings known to be 1 mile apart subtends an angle of 53° at an observer in a balloon, known to be exactly over a point midway between the two buildings. Find the height of the balloon in miles.

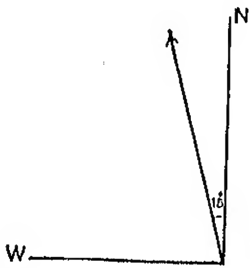
30. The angles of depression and elevation of the top of a tower 50 ft. high, viewed from the top and bottom of a second tower, are 23° and 17°. Find the height of the second tower to the nearest foot.

30. Many problems require a knowledge of the points of the **Compass.**



The angle between any direction and the next is one-eighth of a right-angle, *i.e.* 11° 15′.

The notation 15° West of North means the direction shown in the figure. It is sometimes written N. 15° W.



EXAMPLES XI.

1. A man walks 3 kilometres N.E. and 6 kilometres N.W., how far is he from the starting place to the nearest hectometre?

2. A man walking due West observes two forts due South of him, after walking 6 miles the forts bear 50° and 55° South of East. Find the man's distance from each fort at the two places of observation.

3. A and B start walking in directions N. 17° W. and N. 73° E.: find their distance apart after two hours and the direction of the line joining them. A walks 3 miles an hour and B 4 miles per hour.

4. Two forts A and B are built in the sea E. and W. of each other. A ship is South of A and South West of B; after sailing 10 miles E.N.E. she is South of B. How far are the forts apart?

5. From a station A an object B bore S.E. by S. but after the observer had walked 1000 yards S.W. by W. it bore due E. Find AB.

6. A ship A is 10 miles S.W. of a harbour at the instant another ship B is leaving the harbour; B steams S.E. at 8 miles per hour and the ships meet in 2 hours. Find A's course to nearest degree.

7. A, B and C are three places. B is 30 miles E.N.E. of A, and C is 40 miles S.S.E. of B. Find the distance and bearing of C from A.

8. At 2 p.m. on a certain day a ship sailing N. at a uniform rate of 5 knots passed another vessel sailing W. at 12 knots. Find the bearing and distance of one ship from the other at the preceding noon. [1 knot = 6080 ft. per hour.]

9. A man walking along a road which runs S.W. sees an object S. 60° W. of him, after walking 1000 yds. the object is S. 80° W. Find distance of object from the first point of observation and its shortest distance from the road. Answer to the nearest yard.

10. At two points 5 kilometres apart on a road running North and South the bearings of a building are W. 57° N. and W. 43° S. Find to the nearest metre the distance of the building from the road.

11. A and B are two towns on the banks of a straight river. C is a third town. B is 10 miles N. 10° W. of A; C is N. 20° E. of B and N. 10° E. of A. Find distance of C from the river to the nearest tenth.

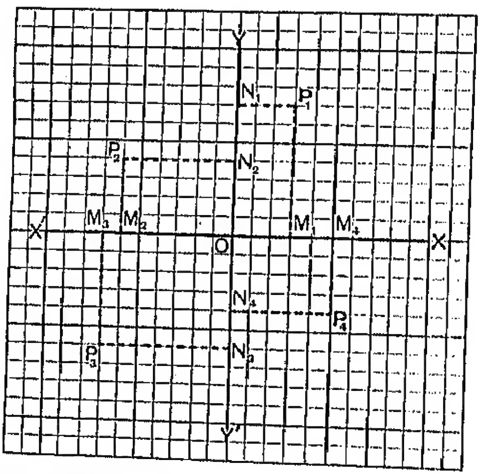
12. A and B are two buoys. B is 2 miles N. 30° E. from A. A third buoy C is due North of A and S. 40° W. of B. Find BC.

CHAPTER VI.

APPLICATIONS OF ALGEBRAIC SIGNS; ANGLES OF ANY MAGNITUDE.

31. Positive and Negative Lines.

The position of a point relatively to two fixed lines XOX′ and YOY′ can conveniently be found by considering



all horizontal distances measured to the right of YOY′ to be positive and those to the left negative, while vertical distances measured upwards from XOX′ are positive and those measured downwards negative.

Thus P_1 is 3 divisions to the *right* and 7 *up* and may be called the point (3, 7).

P_2 is 6 divisions to the *left* and 4 *up* and may be called the point (– 6, 4).

P_3 is 7 divisions to the *left* and 6 *down* and may be called the point (– 7, – 6).

P_4 is 5 divisions to the *right* and 4 *down* and may be called the point (5, – 4).

N_1P_1, OM_1, OM_4, N_4P_4 are positive lines

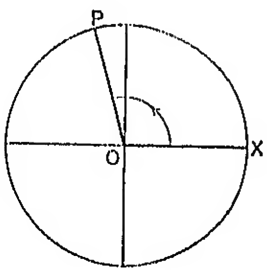
N_2P_2, OM_2, OM_3, N_3P_3 „ negative „

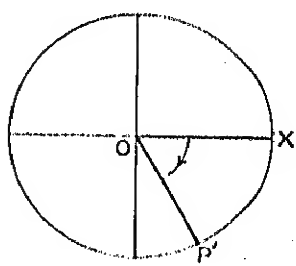
M_1P_1, ON_2, ON_1, M_2P_2 „ positive „

M_4P_4, ON_4, ON_3, M_3P_3 „ negative „

32. Positive and Negative Angles.

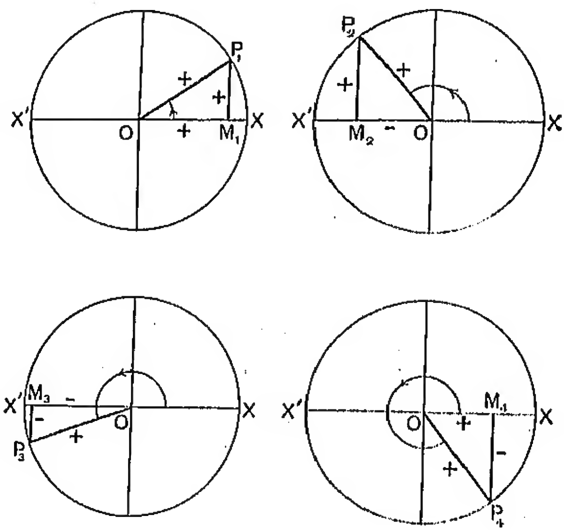
If a straight line starting from OX revolve to OP in a direction contrary to the hands of a clock it is said to describe a positive angle. If it revolve to OP′ in the same direction as the hands of a clock it is said to describe a negative angle.





TRIGONOMETRICAL RATIOS OF ANGLES OF ANY MAGNITUDE.

33. If a line revolving round O from the position OX come into the positions OP_1, OP_2, OP_3, OP_4 and perpendiculars be drawn from P_1, P_2, P_3, P_4 on to OX or OX′, then in every case



the ratio $\frac{MP}{OP}$ is called sin POX

$\frac{OM}{OP}$ „ „ cos POX

$\frac{MP}{OM}$ „ „ tan POX, etc.

If the revolving line has rotated through an angle α, then in the

1st quadrant, $\sin\alpha = \dfrac{M_1P_1}{OP_1} = \dfrac{+\text{ quantity}}{+\text{ quantity}} = +\text{ quantity},$

$\cos\alpha = \dfrac{OM_1}{OP_1} = \dfrac{+\text{ quantity}}{+\text{ quantity}} = +\text{ quantity},$

$\tan\alpha = \dfrac{M_1P_1}{OM_1} = \dfrac{+\text{ quantity}}{+\text{ quantity}} = +\text{ quantity, etc.}$

2nd quadrant, $\sin\alpha = \dfrac{M_2P_2}{OP_2} = \dfrac{+\text{ quantity}}{+\text{ quantity}} = +\text{ quantity},$

$\cos\alpha = \dfrac{OM_2}{OP_2} = \dfrac{-\text{ quantity}}{+\text{ quantity}} = -\text{ quantity},$

$\tan\alpha = \dfrac{M_2P_2}{OM_2} = \dfrac{+\text{ quantity}}{-\text{ quantity}} = -\text{ quantity, etc.}$

3rd quadrant, $\sin\alpha = \dfrac{M_3P_3}{OP_3} = \dfrac{-\text{ quantity}}{+\text{ quantity}} = -\text{ quantity},$

$\cos\alpha = \dfrac{OM_3}{OP_3} = \dfrac{-\text{ quantity}}{+\text{ quantity}} = -\text{ quantity},$

$\tan\alpha = \dfrac{M_3P_3}{OM_3} = \dfrac{-\text{ quantity}}{-\text{ quantity}} = +\text{ quantity, etc.}$

4th quadrant, $\sin\alpha = \dfrac{M_4P_4}{OP_4} = \dfrac{-\text{ quantity}}{+\text{ quantity}} = -\text{ quantity},$

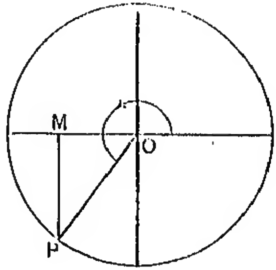
$\cos\alpha = \dfrac{OM_4}{OP_4} = \dfrac{+\text{ quantity}}{+\text{ quantity}} = +\text{ quantity},$

$\tan\alpha = \dfrac{M_4P_4}{OM_4} = \dfrac{-\text{ quantity}}{+\text{ quantity}} = -\text{ quantity, etc.}$

34. These signs of the trigonometrical ratios may be collected in a table as shown; but the student should *not* attempt to commit it to memory.

	Positive	Negative
1st Quadrant	sin cos tan cosec sec cot	
2nd Quadrant	sin cosec	cos sec tan cot
3rd Quadrant	tan cot	sin cosec cos sec
4th Quadrant	cos sec	sin cosec tan cot

35. All the trigonometrical relations already proved are true for angles of any magnitude, positive or negative; for if OP is in the 3rd quadrant for instance,



$$\sin \alpha = \frac{MP}{OP},$$

$$\cos \alpha = \frac{OM}{OP},$$

$$\tan \alpha = \frac{MP}{OM},$$

where to PM, OM, OP are assigned their algebraical values.

$$\therefore \frac{\sin \alpha}{\cos \alpha} = \frac{MP}{OP} \Big/ \frac{OM}{OP} = \frac{MP}{OM} = \tan \alpha.$$

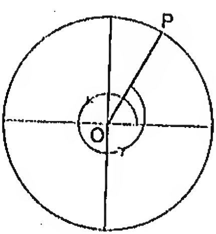
Similarly it may be proved that

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

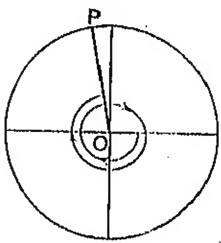
$$1 + \tan^2 \alpha = \sec^2 \alpha, \text{ etc.}$$

Ex. 1. Draw a diagram showing in which quadrant the revolving line lies for an angle of 420°.

$$420° = 360° + 60°.$$



Ex. 1.



Ex. 2.

Ex. 2. Show by a diagram the position of the revolving line for an angle of $-620°$.

$$-620° = -360° - 260°.$$

EXAMPLES XII.

Draw diagrams showing in which quadrant the revolving line lies in each of the following angles:

1. $120°, 225°$. 2. $240°, 150°$. 3. $300°, -300°$.

4. $135°, 325°$. 5. $-150°, 380°$. 6. $-330°, -850°$.

7. $775°, -\frac{3\pi}{4}$. 8. $-725°, \frac{3\pi}{5}$. 9. $225°, -1000°$.

Find the algebraic signs which must be attached to the values of the sine and tangent of

10. $135°, 225°$. 11. $240°, -300°$. 12. $150°, -225°$.

13. $210°, 315°$. 14. $325°, 570°$. 15. $\frac{3\pi}{4}, -\frac{5\pi}{8}$.

16. $1000°, -750°$. 17. $-880°, 335°$. 18. $\frac{7\pi}{8}, -\frac{11\pi}{4}$.

Find the algebraic signs which must be given to the values of the cosine and cosecant of

19. $-135°, 150°$. 20. $225°, -210°$. 21. $300°, 240°$.

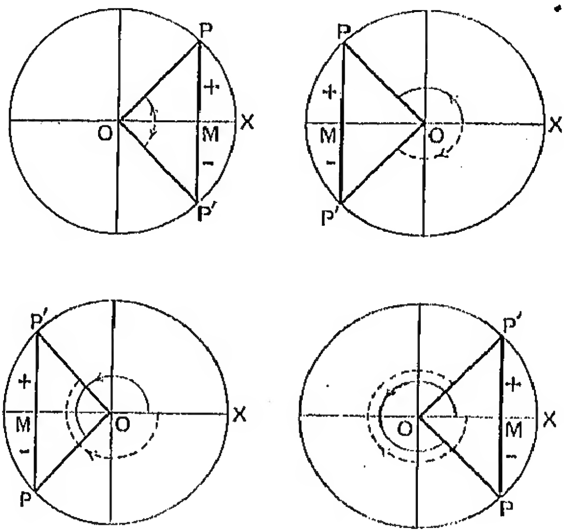
22. $175°, -575°$. 23. $\frac{11\pi}{12}, -\frac{5\pi}{6}$. 24. $-2000°, 3180°$.

36. Trigonometrical ratios of $-\alpha$ for all values of α.

Let two lines starting from OX revolve, one through an angle α to the position OP, and the other through an angle $-\alpha$ to the position OP′.

In all the quadrants it is obvious that $P\hat{O}M = P'\hat{O}M$ and $\therefore \triangle OPM = \triangle OP'M$ in all respects.

$\therefore$ $MP = MP'$ (numerically) $= -MP'$ (algebraically)



$$\sin(-\alpha) = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\alpha,$$

$$\cos(-\alpha) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\alpha,$$

$$\tan(-\alpha) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\alpha,$$

$$\operatorname{cosec}(-\alpha) = \frac{OP'}{MP'} = \frac{OP}{-MP} = -\operatorname{cosec}\alpha,$$

$$\sec(-\alpha) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\alpha,$$

$$\cot(-\alpha) = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\alpha.$$

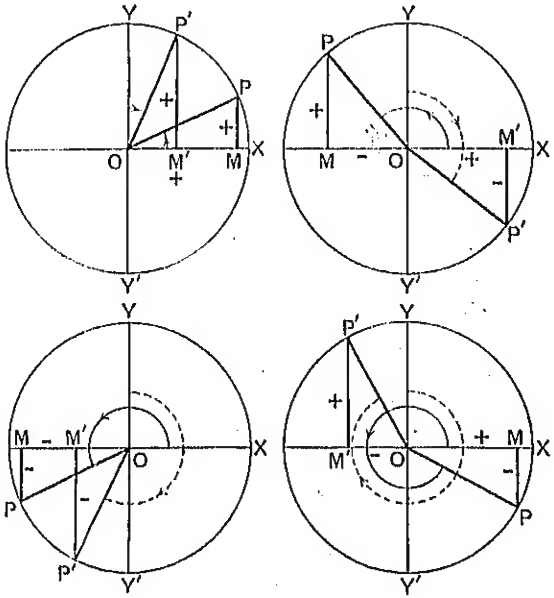
These last four values might have been deduced from the first two.

It will be observed that the only two ratios which remain unchanged in sign are $\cos(-\alpha)$ and $\sec(-\alpha)$.

Jr. Inter March 1943

37. Trigonometrical ratios of $(90° - \alpha)$ for all values of α.

Let two lines starting from OX revolve, one through an angle α to the position OP, and the other through an angle $(90° - \alpha)$ to the position OP′.



OP′ is obtained by revolving through 90° from OX to OY and then negatively through α to the position OP′.

Thus the negative revolution from OY to OP′ is in all the figures equal in magnitude to the positive revolution from OX to OP.

It is obvious from the diagrams, that when OP is in the first and fourth quadrants, $P\hat{O}M = P'\hat{O}Y$.

But $$P'\hat{O}Y = O\hat{P'}M',$$

$$\therefore\ P\hat{O}M = O\hat{P'}M'.$$

Similarly in the second and third quadrants,

$$P\hat{O}M = P'\hat{O}Y'.$$

But $$P'\hat{O}Y' = O\hat{P}'M',$$

$$\therefore\ P\hat{O}M = O\hat{P}'M',$$

$\therefore$ in all the diagrams the Δs OPM and P'OM' are equal in all respects, and

$$MP = OM' \text{ (algebraically)},$$

$$OM = M'P' \text{ (algebraically)},$$

$$\sin(90^\circ - \alpha) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\alpha,$$

$$\cos(90^\circ - \alpha) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\alpha,$$

$$\tan(90^\circ - \alpha) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\alpha.$$

Similarly $$\operatorname{cosec}(90^\circ - \alpha) = \sec\alpha,$$

$$\sec(90^\circ - \alpha) = \operatorname{cosec}\alpha,$$

$$\cot(90^\circ - \alpha) = \tan\alpha.$$

38. Trigonometrical ratios of $(90^\circ + \alpha)$ for all values of α.

Let two lines starting from OX revolve, one through an angle α to the position OP, and the other through an angle $(90^\circ + \alpha)$ to the position OP'.

Then for all positions of OP,

$$P\hat{O}P' = 90^\circ,$$

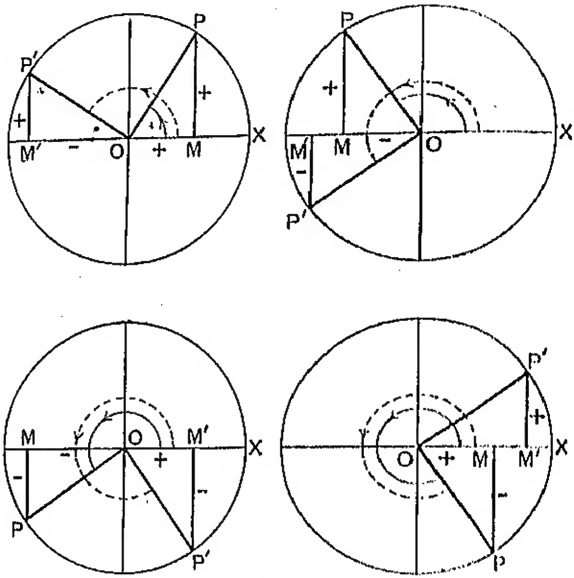
$$\therefore\ P\hat{O}M = \text{complement of } P'\hat{O}M'$$

$$= O\hat{P}'M',$$

$\therefore$ Δs OMP and P′M′O are equal in all respects,

$\therefore$ M′P′ = OM (numerically) = OM (algebraically),

OM′ = MP (numerically) = − MP (algebraically).



$$\therefore \sin(90^\circ+\alpha)=\frac{M'P'}{OP'}=\frac{OM}{OP}=\cos\alpha,$$

$$\cos(90^\circ+\alpha)=\frac{OM'}{OP'}=\frac{-MP}{OP}=-\sin\alpha,$$

$$\tan(90^\circ+\alpha)=\frac{M'P'}{OM'}=\frac{OM}{-MP}=-\cot\alpha.$$

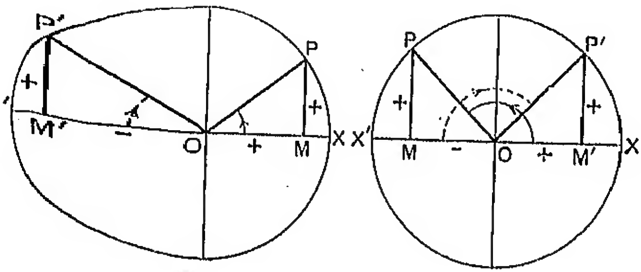
Similarly $\quad \operatorname{cosec}(90^\circ+\alpha)=\sec\alpha,$

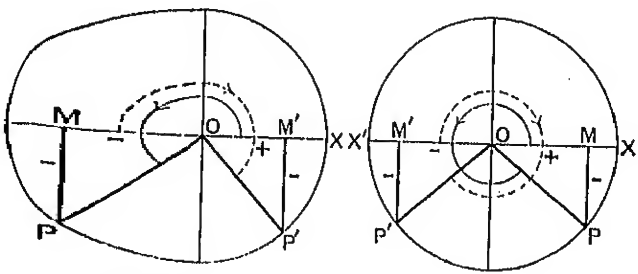
$$\sec(90^\circ+\alpha)=-\operatorname{cosec}\alpha,$$

$$\cot(90^\circ+\alpha)=-\tan\alpha.$$

9. Trigonometrical ratios of $(180° - \alpha)$ for all values of α.

Let two lines starting from OX revolve, one through an angle α to the position OP, and the other through an angle $(180° - \alpha)$ to the position OP′.





OP′ is obtained by revolving through 180° from OX to OX′ in a positive direction and then negatively through α to the position OP′.

Since this negative revolution from OX′ to OP′ is in all the figures equal in magnitude to the positive revolution from OX to OP, it follows that

$$P\hat{O}M = P'\hat{O}M',$$

∴ △s OPM and OP′M′ are equal in all respects.

∴ M′P′ = MP (algebraically),

and OM′ = OM (numerically) = − OM (algebraically).

$$\sin(180^\circ - \alpha) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\alpha,$$

$$\cos(180^\circ - \alpha) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\alpha,$$

$$\tan(180^\circ - \alpha) = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan\alpha.$$

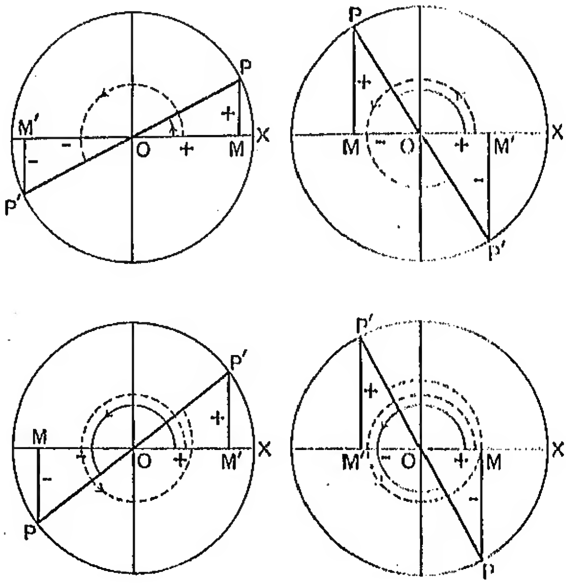
Similarly $\quad \operatorname{cosec}(180^\circ - \alpha) = \operatorname{cosec}\alpha,$

$\quad\quad\quad\quad \sec(180^\circ - \alpha) = -\sec\alpha,$

$\quad\quad\quad\quad \cot(180^\circ - \alpha) = -\cot\alpha.$

40. Trigonometrical ratios of $(180^\circ + \alpha)$ for all values of α.

Let two lines starting from OX revolve, one through an angle α to the position OP, and the other through an angle $(180^\circ + \alpha)$ to the position OP′.



Since the difference between these two angles is 180°, it follows that OP and OP′ are in the same straight line,

$$\therefore P\hat{O}M = P'\hat{O}M'$$

and $\triangle$s POM and P′OM′ are equal in all respects.

$$M'P' = MP \text{ (numerically)} = -MP \text{ (algebraically)},$$

$$OM' = OM \text{ (numerically)} = -OM \text{ (algebraically)}.$$

$$\sin(180° + \alpha) = \frac{M'P'}{OP'} = \frac{-MP}{OP} = -\sin\alpha,$$

$$\cos(180° + \alpha) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\alpha,$$

$$\tan(180° + \alpha) = \frac{M'P'}{OM'} = \frac{-MP}{-OM} = \tan\alpha.$$

Similarly $\operatorname{cosec}(180° + \alpha) = -\operatorname{cosec}\alpha$,

$$\sec(180° + \alpha) = -\sec\alpha,$$

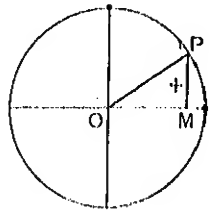
$$\cot(180° + \alpha) = \cot\alpha.$$

The Trigonometrical Ratios of 270° + A can be worked out by similar methods.

41. The cases where the angle α is greater than 360° introduce no difficulties, since the addition or subtraction of any multiple of 360° to the angle α, does not affect the final position of the line OP.

Thus the trigonometrical ratios of $n.360° + \alpha$ will be identical with those of α and the ratios of $n.360° - \alpha$ the same as those of $(-\alpha)$, where n is any integer, positive or negative.

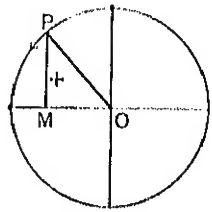
42. To find the variation in value of $\sin\alpha$ as α increases.



As the angle increases from 0° to 90°, MP is positive and increases in magnitude from 0 to OP, OP remaining constant,

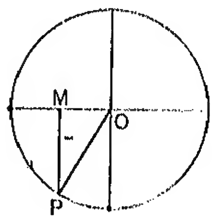
$$\sin\alpha = \frac{MP}{OP};$$

$\therefore$ $\sin\alpha$ is *positive* and *increases* from 0 to 1.



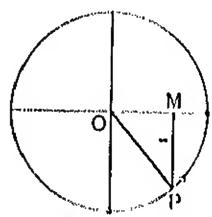
From 90° to 180°, MP is positive and decreases in magnitude from OP to 0,

$\therefore$ $\sin\alpha$ is *positive* and *decreases* from 1 to 0.



From 180° to 270°, MP is negative and increases numerically from 0 to OP,

$\therefore$ $\sin\alpha$ is *negative* and *decreases* from 0 to -1.



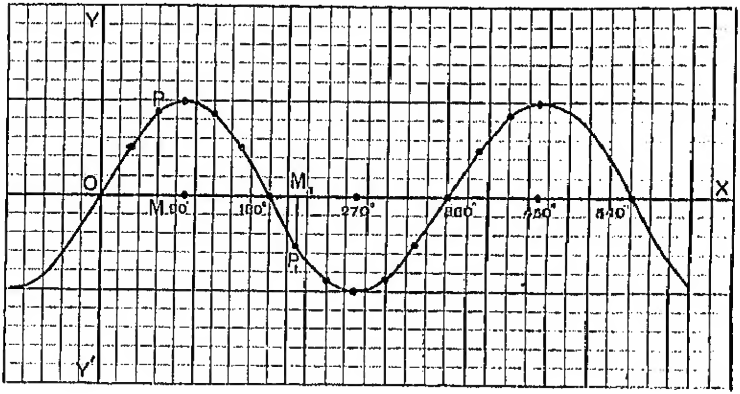
From 270° to 360°, MP is negative and decreases numerically from OP to 0,

$\therefore$ $\sin\alpha$ is *negative* and *increases* from -1 to 0.

Between 360° and 450° we have a repetition of the changes from 0° to 90°, and so on for each successive quadrant; the changes between 360° and 720°, 720° and 1080° etc. being a repetition of the changes between 0° and 360°.

Since the values of $\sin\alpha$ are constantly being repeated in the same order, they are said to be *Periodic Functions*, the *period* being 360° or 2π.

43. This variation in the value of $\sin \alpha$ may be conveniently represented by means of a graph.



Let each division measured along OX represent 20°, and each division along OY represent ·2, then from the known values of the trigonometrical ratios of 0°, 30°, 60°, 90° we have

α	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin \alpha$	0	·5	·9	1	·9	·5	0	− ·5	− ·9	− 1	− ·9	− ·5	0

From these values, taking distances along OX to represent the angle α and distances parallel to OY the value of $\sin \alpha$, the graph may be plotted; for instance,

$$OM = 3 \text{ divisions} = 60°$$

and

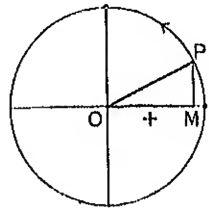
$$MP = 4{\cdot}5 \text{ divisions} = {\cdot}9,$$

$$OM_1 = 10{\cdot}5 \text{ divisions} = 210°,$$

$$M_1P_1 = -\,2{\cdot}5 \text{ divisions} = -\,{\cdot}5.$$

44. To find the variation in value of cos α as α increases.

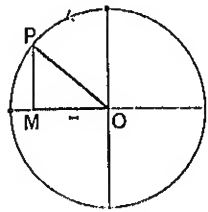
As the angle increases from 0° to 90°, OM is positive and decreases in magnitude from OP to 0, OP remaining constant,



$$\cos\alpha = \frac{OM}{OP};$$

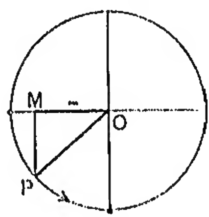
$\therefore$ cos α is *positive* and *decreases* from 1 to 0.

From 90° to 180°, OM is negative and increases numerically from 0 to OP,



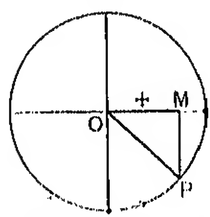
$\therefore$ cos α is *negative* and *decreases* from 0 to – 1.

From 180° to 270°, OM is negative and decreases numerically from OP to 0,



$\therefore$ cos α is *negative* and *increases* from – 1 to 0.

From 270° to 360°, OM is positive and increases in magnitude from 0 to OP,

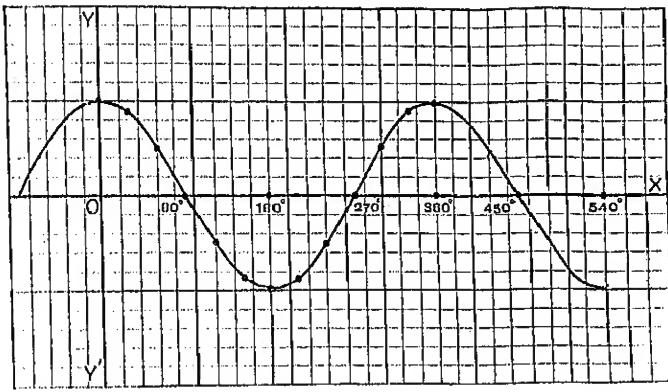


$\therefore$ cos α is *positive* and *increases* from 0 to 1.

Between 360° and 450° we have a repetition of the changes from 0° to 90°, and so on for each successive quadrant, the changes between 360° and 720°, 720° and 1080°, etc. being a repetition of the changes between 0° and 360°.

Thus cos α is also a *Periodic Function*, its period being 360° or 2π.

45. The variation of the value of $\cos\alpha$ may be represented by a graph similar to that of $\sin\alpha$, shifted to the left through 90°.

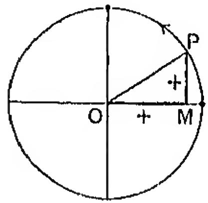


Taking as before each division along OX to represent 20° and each division along OY as ·2, we can trace the graph of $\cos\alpha$ from the following table:

α	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\alpha$	1	·9	·5	0	−·5	−·9	−1	−·9	−·5	0	·5	·9	1

46. To find the variation in the value of tan α as α increases.

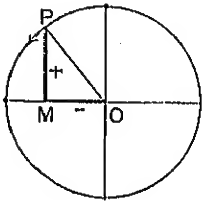
As the angle increases from 0° to 90°, OM is always positive and decreases in magnitude from OP to 0, and MP is positive, increasing from 0 to OP.



$$\tan\alpha = \frac{MP}{OM};$$

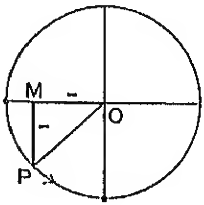
∴ tan α is *positive* and *increases* from 0 to ∞.

From 90° to 180°, MP is positive and decreases from OP to 0, while OM is negative and increases numerically from 0 to OP,



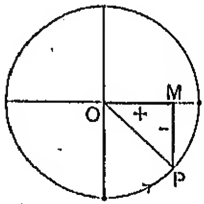
∴ tan α is *negative* and *increases* from $-\infty$ to 0.

From 180° to 270°, MP is negative and increases numerically from 0 to OP, and OM is also negative and decreases numerically from OP to 0,



∴ tan α is *positive* and *increases* from 0 to ∞.

From 270° to 360°, MP is negative while OM is positive, MP decreasing numerically from OP to 0, and OM increasing from 0 to OP,



∴ tan α is *negative* and *increases* from $-\infty$ to 0.

Between 360° and 720°, we again have a repetition of the values between 0° and 360°, and so on for each cycle.

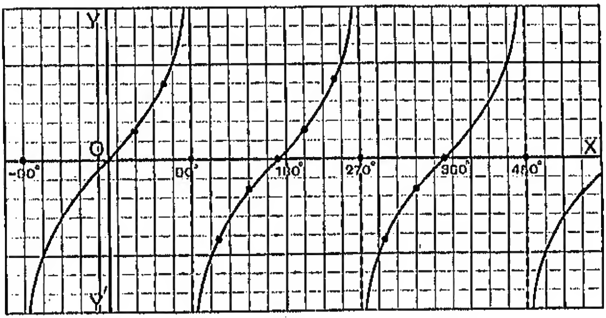
Thus tan α is a *Periodic Function*, its *period* being 360° or 2π.

47. The variation in the value of $\tan\alpha$ may be shown graphically.

From the following table, giving the connection between α and $\tan\alpha$, the curve shown in the diagram may be plotted.

α	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\tan\alpha$	0	·6	1·7	∞	−1·7	−·6	0	·6	1·7	∞	−1·7	−·6	0

Each division along OX represents 20° and each division along OY represents ·4. It will be seen that the various portions of the curve approach the vertical lines through 90°, 270°, 450°, etc., and eventually touch them when the value of $\tan\alpha$ becomes either $\pm\infty$.



48. The variation of cosec α.

With diagrams similar to those given in Art. 42, since $\operatorname{cosec}\alpha = \frac{OP}{MP}$, it follows that cosec α is positive and varies from $\frac{OP}{0}$ to $\frac{OP}{OP}$ as α increases from 0° to 90°,

i.e. cosec α is *positive* and *decreases* from ∞ to 1.

Between 90° and 180°, cosec α is *positive* and *increases* from $\frac{OP}{OP}$ to $\frac{OP}{0}$, *i.e.* from 1 to ∞.

Between 180° and 270°, cosec α is *negative* and *increases* from $\frac{OP}{0}$ to $\frac{OP}{-OP}$, *i.e.* from $-\infty$ to -1.

Between 270° and 360°, cosec α is *negative* and *decreases* from $\frac{OP}{-OP}$ to $\frac{OP}{0}$, *i.e.* from -1 to $-\infty$.

49. The connection between cosec α and α is shown in the following table and from the values there given a curve may be plotted.

α	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cosec α	∞	2	1·2	1	1·2	2	∞	−2	−1·2	−1	−1·2	−2	∞

50. The variation of $\sec\alpha$.

With diagrams similar to those given in Art. 44, it follows that since $\sec\alpha = \dfrac{OP}{OM}$, it is *positive* when α is between 0° and 90° and *increases* from $\dfrac{OP}{OP}$ to $\dfrac{OP}{0}$, *i.e.* from 1 to ∞.

Between 90° and 180° it is *negative* and *increases* from $\dfrac{OP}{0}$ to $\dfrac{OP}{-OP}$, *i.e.* from $-\infty$ to -1.

Between 180° and 270° it is *negative* and *decreases* from $\dfrac{OP}{-OP}$ to $\dfrac{OP}{0}$, *i.e.* from -1 to $-\infty$.

Between 270° and 360° it is *positive* and *decreases* from $\dfrac{OP}{0}$ to $\dfrac{OP}{OP}$, *i.e.* from ∞ to 1.

The curve will be similar to that for $\operatorname{cosec}\alpha$ moved 90° to the left.

51. The variation of $\cot\alpha$.

With the diagrams of Art. 46, it follows that since $\cot\alpha = \dfrac{OM}{MP}$, it is *positive* when α is between 0° and 90° and *decreases* from $\dfrac{OP}{0}$ to $\dfrac{0}{OP}$, *i.e.* from ∞ to 0.

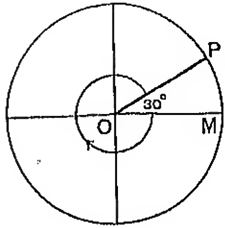
Between 90° and 180° it is *negative* and *decreases* from $\dfrac{0}{OP}$ to $\dfrac{-OP}{0}$, *i.e.* from 0 to $-\infty$.

Between 180° and 270° it is *positive* and *decreases* from $\dfrac{OP}{0}$ to $\dfrac{0}{OP}$, *i.e.* from ∞ to 0.

Between 270° and 360° it is *negative* and *decreases* from $\dfrac{0}{-OP}$ to $\dfrac{OP}{0}$, *i.e.* from 0 to $-\infty$.

Ex. 1. Find the value of

$$\tan(-330°).$$

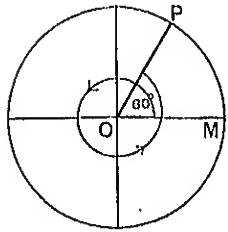


By revolving in a negative direction as shown in the diagram through 330°, OP is in the 1st quadrant and

$$P\hat{O}M = 360° - 330° = 30°$$

$$\therefore \tan(-330°) = \tan 30° = \frac{1}{\sqrt{3}}.$$

Ex. 2. Find the value of sin 420°.

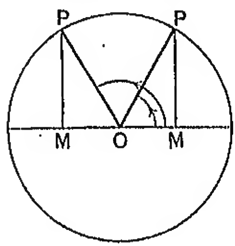


Revolving in a positive direction through 420°, OP is in the 1st quadrant and

$$P\hat{O}M = 60°$$

$$\therefore \sin 420° = \sin 60° = \frac{\sqrt{3}}{2}.$$

Ex. 3. Find all the angles $< 360°$ which satisfy $\sin\theta = \frac{\sqrt{3}}{2}$.



One value of θ is known to be 60°.

Since $\sin\theta = \frac{MP}{OP} = +\frac{\sqrt{3}}{2}$, it follows that MP and OP must have the same sign, and thus the angle can only be in the 1st and 2nd quadrants,

$$\therefore \theta = 60°, \text{ or } 120°.$$

Ex. 4. Reduce to its simplest form

$$\sin(180° - A)\tan(90° + A)\sec(270° + A).$$

Expression $= \sin A \times (-\cot A) \times \operatorname{cosec} A$

$$= -\sin A \cdot \cot A \cdot \frac{1}{\sin A}$$

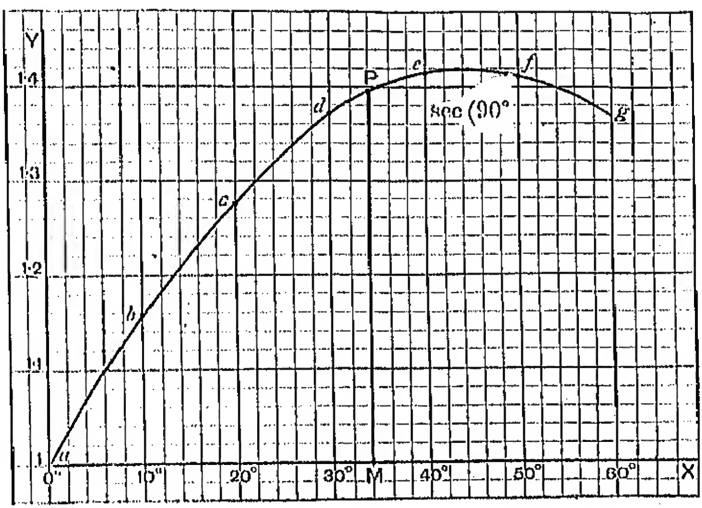
$$= -\cot A.$$

Ex. 5. Find from the tables the values of sin A + cos A when A = 0°, 10°, 20°, 30°, 40°, 50°, 60°. Draw a curve showing how sin A + cos A varies as A increases from 0° to 60°, and thence find its value when A = 34°.

A	sin A + cos A
0°	0 + 1 = 1
10°	·1736 + ·9848 = 1·16 (correct to 2 places)
20°	·3420 + ·9397 = 1·28
30°	·5000 + ·8660 = 1·37
40°	·6428 + ·7660 = 1·41
50°	·7660 + ·6428 = 1·41
60°	·8660 + ·5000 = 1·37

Let distances measured along OX represent degrees.
" " " " OY " values of sin A + cos A.

From the above table, plot the points *a*, *b*, *c*, *d*, *e*, *f*, *g*, and thence draw the graph.



To find the value of sin A + cos A when A = 34°, take OM = 34° and read off the distance MP on the vertical line through M by means of the scale on OY.

We find that sin 34° + cos 34° = 1·39.

EXAMPLES XIII.

Find the values of

1. $\sin 135°$.
2. $\cos 225°$.
3. $\tan 120°$.
4. $\cot 210°$.
5. $\sec 225°$.
6. $\operatorname{cosec} 150°$.
7. $\sin 315°$.
8. $\cos 330°$.
9. $\sin (-30°)$.
10. $\cos (-135°)$.
11. $\tan 240°$.
12. $\cot 225°$.
13. $\sec (-330°)$.
14. $\cot (-300°)$.
15. $\operatorname{cosec} (-330°)$.
16. $\sin \frac{3\pi}{4}$.
17. $\sec 210°$.
18. $\cos \frac{2\pi}{3}$.
19. $\operatorname{cosec} 300°$.
20. $\sec \frac{5\pi}{6}$.
21. $\operatorname{cosec} \frac{5\pi}{3}$.
22. $\tan \frac{11\pi}{6}$.
23. $\cot \frac{7\pi}{6}$.
24. $\cos \left(-\frac{2\pi}{3}\right)$.
25. $\cot \left(-\frac{3\pi}{4}\right)$.
26. $\tan \left(-\frac{5\pi}{4}\right)$.

Evaluate

27. $\sin 480°$.
28. $\cos 960°$.
29. $\tan (-585°)$.
30. $\cot 690°$.
31. $\sin 495°$.
32. $\operatorname{cosec} (-675°)$.
33. $\sin 930°$.
34. $\cos 945°$.
35. $\cot 1290°$.
36. $\operatorname{cosec} 1380°$.
37. $\tan (-945°)$.

Find all the angles $< 360°$ which satisfy

38. $\cos \theta = \frac{\sqrt{3}}{2}$.
39. $\sin \theta = \frac{1}{2}$.
40. $\tan \theta = 1$.
41. $\cot \theta = -\sqrt{3}$.
42. $\sec \theta = -2$.
43. $\operatorname{cosec} \theta = -\sqrt{2}$.

44. If A is between 180° and 270° and $\tan A = \frac{3}{4}$, find the values of $\sin A$ and $\sec A$.

45. If $\cos A = \frac{4}{5}$ and A is between 270° and 360°, find the values of $\sin A$ and $\tan A$.

46. Find the values of cosec A and cot A, if A is between 270° and 360°, and $\sin A = -\frac{5}{13}$.

47. If $\tan A = -\frac{24}{7}$ and A is between 90° and 180°, find the values of cos A and cosec A.

Find in terms of A, the values of

48. $\sin(270^\circ + A)$.
49. $\cos(270^\circ - A)$.
50. $\tan(270^\circ + A)$.
51. $\operatorname{cosec}(270^\circ - A)$.

Reduce to their simplest forms

52. $\sin(180^\circ + A)\cos(90^\circ - A)$.

53. $\cos(180^\circ - A)\cot(90^\circ + A)$.

54. $\cot(180^\circ + A)\sec(180^\circ - A)$.

55. $\tan(90^\circ - A)\operatorname{cosec}(90^\circ + A)$.

56. $\operatorname{cosec}(180^\circ - A)\sec(90^\circ + A)\cot(90^\circ - A)$.

57. $\cot(90^\circ + A)\tan(180^\circ + A)\sec(90^\circ - A)$.

58. $\operatorname{cosec}(180^\circ - A)\sec(180^\circ + A)\tan(90^\circ - A)$.

Prove that

59. $\tan(180^\circ - A) + \tan(180^\circ + A) - \sec(90^\circ + A) = \operatorname{cosec}(360^\circ + A)$.

60. $\cos(90^\circ + A) - \cot(270^\circ + A) - \sin(180^\circ + A) = \tan A$.

61. $\sec(360^\circ - A) + \operatorname{cosec}(270^\circ + A) - \operatorname{cosec}(90^\circ + A) = \sec(180^\circ - A)$.

62. $\cot(90^\circ - A) - \sin A + \cot(90^\circ + A) = \sin(360^\circ - A)$.

63. $-\sin 480^\circ \cos 120^\circ + \cos 240^\circ \sin 120^\circ = 0$.

64. $\cos 150^\circ \cos 420^\circ + \sin 330^\circ \sin 300^\circ = 0$.

65. $\sin 780^\circ \sin 120^\circ + \cos 120^\circ \sin 390^\circ = \frac{1}{2}$.

66. $\sin 600^\circ \cos 330^\circ + \cos 120^\circ \sin 150^\circ = -1$.

What is the value of 2 sin A cos A when

67. $A = \frac{\pi}{2}$.
68. $A = \frac{\pi}{3}$.
69. $A = 120^\circ$.
70. $A = \frac{3\pi}{4}$?

What is the value of $\cos^2 A - \sin^2 A$ when

71. $A = \frac{\pi}{6}$. 72. $4A = \pi$.

73. $A = \frac{5\pi}{6}$. 74. $A = 2\pi$?

75. Find from the tables the values of $\sin A - \cos A$ when $A = 0°, 10°, 20°, 30°, 40°, 50°$. Draw a curve showing how $\sin A - \cos A$ varies as A increases from $0°$ to $50°$ and thence find its value when $A = 26°$. Verify by means of the tables.

76. Use the tables to find the values of $\tan A + \cot A$ (correct to 2 places of decimals) when $A = 0°, 10°, 20°, 30°, 40°, 50°$. Draw a curve showing the variation of $\tan A + \cot A$ and thence find its value when $A = 12°$.

77. In a certain tangent galvanometer, $E = 1{\cdot}91 \tan \delta$, where E is the Electromotive force of the cell and δ is the deflection in degrees. Assuming that the electromotive force is made to vary, illustrate this variation by a graph taking for δ the values $5°$, $10°, 15°, 20°, 25°$. Find the value of E when $\delta = 23°$, and then verify by means of tables. (Answer correct to two places of decimals.)

78. If a particle is projected with a velocity of 64 feet per second at an angle α with the horizontal, the time, in seconds, before it reaches the ground again is given by $t = 4 \sin \alpha$. Calculate the value of this from the tables when $\alpha = 14°, 18°, 22°$, $26°, 30°$ respectively. Draw a curve and thence find the time when $\alpha = 23°$.

Miscellaneous Examples on Chapters V and VI start on page 212, Test Paper XVII.

CHAPTER VII.

LOGARITHMS.

✓

52. IF $a^x = N$, then x is called the logarithm of N to the base a, and the equation may be written

$$x = \log_a N.$$

DEF. *The logarithm of a number to a given base is the index of the power to which the base must be raised to equal the number.*

$$4^3 = 64 \qquad \therefore \log_4 64 \quad = 3$$

$$7^4 = 2401 \qquad \therefore \log_7 2401 = 4.$$

Since $\qquad a^0 = 1 \qquad \therefore \log_a 1 \quad = 0,$

$\therefore$ *the logarithm of* 1 *to any base is* 0.

53. For most practical purposes the base chosen is 10, and the logarithms are then called Common Logarithms; this system was introduced by Briggs in 1615. In writing down such logarithms the base is omitted, so that $\log_{10} 12$ is written $\log 12$.

54. In many theoretical calculations the base used is the infinite series $1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \ldots$ which equals 2·7183 (approx.) and is denoted by e. Such logarithms are known as Napierian or Hyperbolic Logarithms.

Unless otherwise stated we shall deal with Common Logarithms.

55. An easy method of calculating logarithms approximately has been suggested independently by Prof. Perry and Mr Edser.

The square root of 10 is extracted by Arithmetic; then the square root of the answer; then the square root of the new answer, and so on.

Thus

$$\begin{array}{lll}
10^1 = 10 & \therefore & \log 10 = 1, \\
10^{\frac{1}{2}} = 3{\cdot}162 & & \log 3{\cdot}162 = \tfrac{1}{2} = {\cdot}5000, \\
10^{\frac{1}{4}} = 1{\cdot}778 & & \log 1{\cdot}778 = \tfrac{1}{4} = {\cdot}2500, \\
10^{\frac{1}{8}} = 1{\cdot}333 & & \log 1{\cdot}333 = \tfrac{1}{8} = {\cdot}1250, \\
10^{\frac{1}{16}} = 1{\cdot}155 & & \log 1{\cdot}155 = \tfrac{1}{16} = {\cdot}0625, \\
10^{\frac{1}{32}} = 1{\cdot}075 \text{ etc.} & & \log 1{\cdot}075 = \tfrac{1}{32} = {\cdot}0313 \text{ etc.}
\end{array}$$

From these we can deduce other values; for

$$\begin{array}{ll}
10^{\frac{3}{4}} = 10^{\frac{1}{2}} \,.\, 10^{\frac{1}{4}} = 3{\cdot}162 \times 1{\cdot}778 = 5{\cdot}622 & \therefore \log 5{\cdot}622 = {\cdot}7500, \\
10^{\frac{3}{8}} = 10^{\frac{1}{4}} \,.\, 10^{\frac{1}{8}} = 1{\cdot}778 \times 1{\cdot}333 = 2{\cdot}370 & \log 2{\cdot}370 = {\cdot}3750, \\
10^{\frac{5}{8}} = 10^{\frac{1}{2}} \,.\, 10^{\frac{1}{8}} = 3{\cdot}162 \times 1{\cdot}333 = 4{\cdot}215 & \log 4{\cdot}215 = {\cdot}6250, \\
10^{\frac{7}{8}} = 10^{\frac{3}{4}} \,.\, 10^{\frac{1}{8}} = 5{\cdot}622 \times 1{\cdot}333 = 7{\cdot}494 & \log 7{\cdot}494 = {\cdot}8750, \\
10^{\frac{3}{16}} = 10^{\frac{1}{8}} \,.\, 10^{\frac{1}{16}} = 1{\cdot}333 \times 1{\cdot}155 = 1{\cdot}540 & \log 1{\cdot}540 = {\cdot}1875, \\
10^{\frac{5}{16}} = 10^{\frac{1}{4}} \,.\, 10^{\frac{1}{16}} = 1{\cdot}778 \times 1{\cdot}155 = 2{\cdot}054 & \log 2{\cdot}054 = {\cdot}3125, \\
10^{\frac{7}{16}} = \text{etc.} & \\
10^{\frac{3}{32}} = 10^{\frac{1}{16}} \,.\, 10^{\frac{1}{32}} = 1{\cdot}155 \times 1{\cdot}075 = 1{\cdot}242 & \log 1{\cdot}242 = {\cdot}0938, \\
10^{\frac{5}{32}} = 10^{\frac{1}{8}} \,.\, 10^{\frac{1}{32}} = 1{\cdot}333 \times 1{\cdot}075 = 1{\cdot}433 & \log 1{\cdot}433 = {\cdot}1563, \\
10^{\frac{7}{32}} = \text{etc.} &
\end{array}$$

When arranged in order of magnitude, we have the following Table, which of course might contain many intermediate values.

Number	Logarithm	Number	Logarithm
1·000	0	2·054	·3125
1·075	·0313	2·370	·3750
1·155	·0625	3·162	·5000
1·242	·0938	4·215	·6250
1·333	·1250	5·622	·7500
1·433	·1563	7·494	·8750
1·540	·1875	10·000	1·0000
1·778	·2500		

Taking now any three values fairly close together, let us say

Number	Logarithm
1·433	·1563
1·540	·1875
1·778	·2500

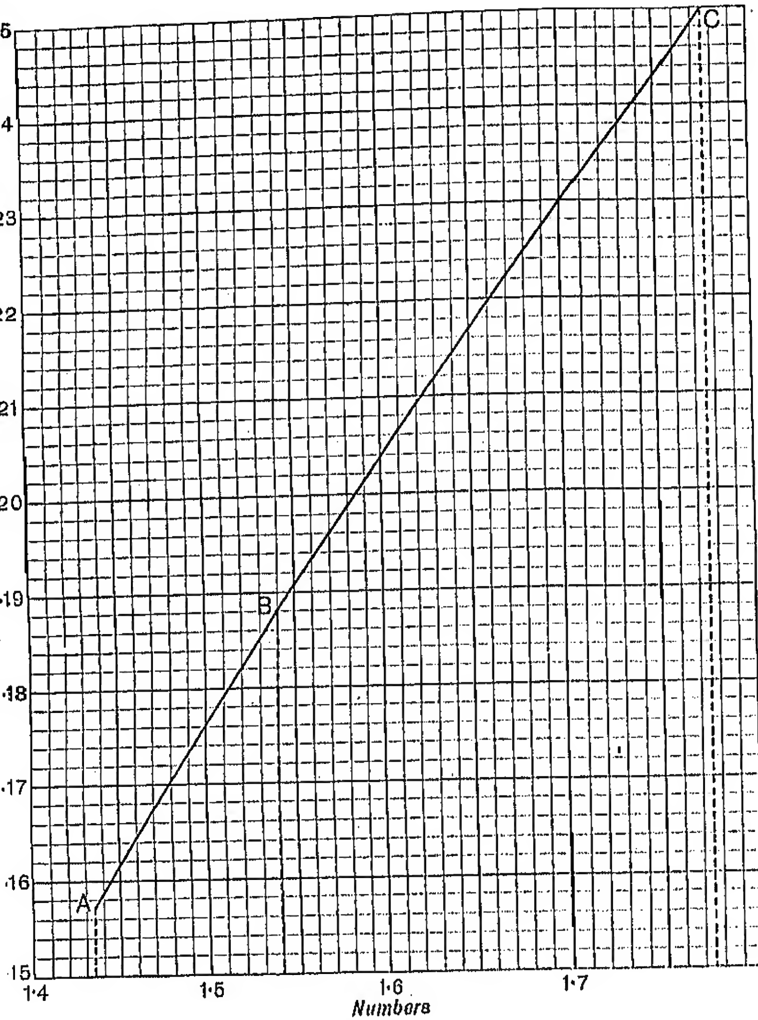
we can plot these on squared paper by the points A, B, C and drawing the straight lines AB, BC, read off the logarithm of any number between 1·433 and 1·778.

The larger the diagram the more accurate will be the answer. From the diagram, we find that

$$\log 1{\cdot}560 = {\cdot}193,$$

$$\log 1{\cdot}650 = {\cdot}217,$$

$$\log 1{\cdot}730 = {\cdot}238 \text{ etc.}$$


·25
·24
·23
·22
·21
·20
·19
·18
·17
·16
·15
A
B
C
1·4
1·5
1·6
1·7
Numbers

56. Theorem I. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.*

Let $\log m = x, \qquad \therefore m = 10^x,$

$\log n = y, \qquad \therefore n = 10^y,$

$$\therefore mn = 10^x . 10^y = 10^{x+y},$$

$$\therefore \log (mn) = x + y$$

$$= \log m + \log n.$$

This theorem may be extended to the product of any number of factors, thus

$$\log (mnp) = \log m + \log n + \log p.$$

57. Theorem II. *The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.*

Let $\log m = x \qquad \therefore m = 10^x,$

$\log n = y \qquad \therefore n = 10^y,$

$$\therefore \frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y},$$

$$\therefore \log \left(\frac{m}{n}\right) = x - y$$

$$= \log m - \log n.$$

58. Theorem III. *The logarithm of the power of any number is equal to the logarithm of the number multiplied by the index of the power.*

Let $\log m = x,$

$$\therefore m = 10^x,$$

$$m^n = 10^{nx},$$

$$\therefore \log (m^n) = nx$$

$$= n \log m.$$

These theorems have been proved for the base 10, but they would apply equally to any other base.

Given that $\log 7{\cdot}211 = {\cdot}8580$ and $\log 8{\cdot}878 = {\cdot}9483$,

Ex. 1. $\log(7{\cdot}211 \times 8{\cdot}878) = \log 7{\cdot}211 + \log 8{\cdot}878 = {\cdot}8580$
$$+{\cdot}9483$$
$$= 1{\cdot}8063.$$

Ex. 2.
$$\log\frac{8{\cdot}878}{7{\cdot}211} = \log 8{\cdot}878 - \log 7{\cdot}211$$
$$= {\cdot}9483$$
$$-{\cdot}8580$$
$$= {\cdot}0903.$$

Ex. 3.
$$\log(7{\cdot}211)^5 = 5\log 7{\cdot}211$$
$$= 5 \times {\cdot}8580$$
$$= 4{\cdot}2900.$$

Ex. 4.
$$\log\sqrt[3]{7{\cdot}211} = \tfrac{1}{3}\log 7{\cdot}211$$
$$= \tfrac{1}{3} \times {\cdot}8580$$
$$= {\cdot}2860.$$

59. If a logarithm is partly integral and partly fractional, then the integral part is called the *Characteristic* and the fractional part the *Mantissa*.

It is always so arranged that the mantissa is positive.

Thus
$$\log\tfrac{1}{4} = \log 1 - \log 4$$
$$= 0 - {\cdot}6021 \text{ (from Tables)}$$
$$= -1 + 1 - {\cdot}6021$$
$$= -1 + {\cdot}3979$$
$$= \bar{1}{\cdot}3979.$$

If it is necessary to divide such a logarithm by a number, the negative characteristic is increased until it is a multiple of the divisor, compensation being made by adding the necessary positive integer.

Ex. Given that $\log {\cdot}03 = \bar{2}{\cdot}4771$, find the value of $\log({\cdot}03)^{\frac{1}{3}}$.
$$\log({\cdot}03)^{\frac{1}{3}} = \tfrac{1}{3}\log{\cdot}03 = \tfrac{1}{3}(\bar{2}{\cdot}4771)$$
$$= \tfrac{1}{3}(\bar{3} + 1{\cdot}4771)$$
$$= \bar{1}{\cdot}4924.$$

60.

$$\begin{aligned}
10^3 &= 1000 & \therefore \log 1000 &= 3,\\
10^2 &= 100 & \log 100 &= 2,\\
10^1 &= 10 & \log 10 &= 1,\\
10^0 &= 1 & \log 1 &= 0,\\
10^{-1} &= \cdot 1 & \log \cdot 1 &= -1,\\
10^{-2} &= \cdot 01 & \log \cdot 01 &= -2,\\
10^{-3} &= \cdot 001 & \log \cdot 001 &= -3.
\end{aligned}$$

It is therefore seen that the

Logarithm of a number between 100 and 1000, *i.e.* with **3** digits = **2** + fraction.

Logarithm of a number between 10 and 100, *i.e.* with **2** digits = **1** + fraction.

Logarithm of a number between 1 and 10, *i.e.* with **1** digit = **0** + fraction.

Logarithm of a number between ·1 and 1 = $\bar{\mathbf{1}}$ + fraction.

Logarithm of a number between ·**0**1 and ·1 = $\bar{\mathbf{2}}$ + fraction.

Logarithm of a number between ·**00**1 and ·01 = $\bar{\mathbf{3}}$ + fraction.

Thus the characteristic of the logarithm of a number can be written down by inspection.

RULE. *The characteristic of the logarithm of a number* > 1 *is positive and is one less than the number of digits before the decimal point.*

The characteristic of a number < 1 *is negative and is one more than the number of ciphers immediately after the decimal point.*

61. Without giving any rule, the **characteristic can at once be determined**, by writing the number as the product of a number between 1 and 10, and some multiple of 10; the *characteristic* is then the *same as the index* of the power of 10.

Thus

$$\log 7412 = \log(7{\cdot}412\times 10^3) = \log 7{\cdot}412 + 3\log 10 = \text{fraction} + 3,$$

$$\log 34{\cdot}12 = \log(3{\cdot}412\times 10) = \log 3{\cdot}412 + \log 10 = \text{fraction} + 1,$$

$$\log {\cdot}7132 = \log(7{\cdot}132\times 10^{-1}) = \log 7{\cdot}132 - \log 10 = \text{fraction} - 1,$$

$$\log {\cdot}00713 = \log(7{\cdot}13\times 10^{-3}) = \log 7{\cdot}13 - 3\log 10 = \text{fraction} - 3.$$

62. *The mantissae of logarithms of all numbers having the same significant figures are the same.*

The truth of this is easily seen by considering a few examples.

Given

$$\log 4{\cdot}63 = {\cdot}6656,$$
$$\log {\cdot}0463 = \log(4{\cdot}63\times 10^{-2}) = {\cdot}6656 - 2 = \bar{2}{\cdot}6656,$$
$$\log 46{\cdot}3 = \log(4{\cdot}63\times 10) = {\cdot}6656 + 1 = 1{\cdot}6656,$$
$$\log 4630 = \log(4{\cdot}63\times 10^3) = {\cdot}6656 + 3 = 3{\cdot}6656.$$

Ex. 1. If $\log 6{\cdot}478 = {\cdot}8114$, what are the logarithms of $64{\cdot}78$, ${\cdot}006478$?

$\log 64{\cdot}78 = 1{\cdot}8114$, by Arts. 60 and 62,

$\log {\cdot}006478 = \bar{3}{\cdot}8114$.

Ex. 2. If $\log 796{\cdot}2 = 2{\cdot}9010$, write down the numbers whose logarithms are ${\cdot}9010$, $\bar{1}{\cdot}9010$, $5{\cdot}9010$, $\bar{4}{\cdot}9010$.

$7{\cdot}962$, ${\cdot}7962$ $(=7{\cdot}962\times 10^{-1})$, 796200 $(=7{\cdot}962\times 10^5)$, ${\cdot}0007962$ $(=7{\cdot}962\times 10^{-4})$.

Ex. 3. Given that $\log 3 = {\cdot}4771$, find the number of digits in 3^{15} and the position of the first significant figure in 3^{-15}.

(i) Let $x = 3^{15}$,

$\therefore\ \log x = 15\log 3 = 7{\cdot}1565$.

Since the characteristic is 7, the number of digits in the integral part of 3^{15} is 8.

(ii) Let $x = 3^{-15}$,

$$\therefore \log x = -15 \log 3 = -7 \cdot 1565$$
$$= \overline{8} \cdot 8435.$$

Therefore the number of ciphers to the right of the decimal point in 3^{-15} is 7, and thus the first significant figure is the 8th.

63. Transformation of logarithms.

Logarithm Tables are constructed by calculating the logarithms to the base e (Art. 54) by means of the series given in Art. 219 and then converting them to the base 10 as follows.

Let $a =$ logarithm of N to base e

$x =$ logarithm of N to base 10.

Then $10^x = \mathrm{N} = e^a$,

and taking logarithms to the base e

$$x \log_e 10 = a$$

$$x = \frac{a}{\log_e 10} = \frac{a}{2 \cdot 30258}$$
$$= a \times \cdot 43429.$$

The general case for the transformation of logarithms is as follows.

Given the value of $\log_a \mathrm{N}$, suppose we wish to find the value of $\log_b \mathrm{N}$.

Let $\log_b \mathrm{N} = x$, $\therefore b^x = \mathrm{N}$,

$$\log_a (b^x) = \log_a \mathrm{N},$$

$$i.e. \quad x \log_a b = \log_a \mathrm{N},$$

$$\therefore \log_b \mathrm{N} = x = \log_a \mathrm{N} \times \frac{1}{\log_a b}.$$

If we put $\mathrm{N} = a$, then

$$\log_b a = 1 \times \frac{1}{\log_a b},$$

or $$\log_b a \times \log_a b = 1.$$

EXAMPLES XIV.

1. What are the characteristics of the logarithms of 527·3, 3·265, ·8275, ·00823, 8134·27, ·000417?

2. If $\log 7645 = 3{\cdot}8834$, write down the logarithms of 76·45, 764·5, ·07645, ·0007645, 76450, 7·645.

3. If $\log 3{\cdot}735 = {\cdot}5723$, write down the numbers whose logarithms are 3·5723, 1·5723, $\bar{2}{\cdot}5723$, $\bar{5}{\cdot}5723$, 5·5723.

4. Given that $\log 2 = {\cdot}3010$, find the number of digits in the integral parts of 2^{25}, 2^{33}, 2^{45}.

5. Given that $\log 3 = {\cdot}4771$, find the position of the first significant figure in 3^{-9}, 3^{-15}, 3^{-21}.

6. Given that $\log 44{\cdot}35 = 1{\cdot}6469$, find the values of

$$\log (44{\cdot}35)^{\frac{1}{2}}, \quad \log (4{\cdot}435)^{\frac{1}{4}}, \quad \log (4435)^{\frac{1}{7}},$$
$$\log ({\cdot}4435)^{2}, \quad \log ({\cdot}04435)^{4}, \quad \log (443{\cdot}5)^{6},$$

correct to 4 places of decimals.

7. Given that $\log 4{\cdot}4 = {\cdot}6435$, find the values of

$$\log (4{\cdot}4)^{\frac{2}{3}}, \quad \log (4{\cdot}4)^{\frac{3}{5}}, \quad \log (440)^{\frac{2}{7}}, \quad \log ({\cdot}44)^{\frac{1}{6}}.$$

8. If

$$\log 32{\cdot}1 = 1{\cdot}5065, \quad \log 4{\cdot}27 = {\cdot}6304, \quad \text{and} \quad \log 848 = 2{\cdot}9284,$$

find the values of

(i) $\log (321 \times 427)$,

(ii) $\log (3{\cdot}21 \times 42{\cdot}7 \times 848)$,

(iii) $\log \dfrac{3{\cdot}21}{42{\cdot}7}$,

(iv) $\log \dfrac{321 \times 84{\cdot}8}{{\cdot}427}$,

(v) $\log \dfrac{{\cdot}0427}{32{\cdot}1 \times {\cdot}0848}$.

THE USE OF LOGARITHM TABLES.

64. We have already seen that the *characteristic* of the logarithm of a number may be written down by inspection, so that the mantissae only are found in the Tables.

To find the mantissa of the logarithm of a number with four significant figures, we firstly look for the *first two* significant figures in the first column, and passing along the row containing these, take the number in that particular column headed by the *third* figure; to this number is added the number in that particular difference column headed by the *fourth* figure.

Ex. To find log 4·257.

We firstly look for the row containing 42 in the first column, and in this row select the number in the column headed by the third figure 5; this gives us 6284. In this same row, the number in the difference column headed by the fourth figure 7 is 7.

$$\therefore \text{ mantissa is } 6291.$$

Thus $$\log 4{\cdot}257 = {\cdot}6291.$$

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9

[If the number whose logarithm is required does not contain 4 significant figures, we can add ciphers until it does. Thus to find log 3, we should look for log 3000. If the number contains more than 4 significant figures, it is written down correct to 4 figures.]

65. The numbers given in the above difference columns are only approximate, and more accurate tables can be constructed by replacing each row in the difference column by two rows, the first of which is used when the third figure of the original number is between 0 and 4 inclusive, and the second when the third figure of the original number is between 5 and 9 inclusive.

Thus

$$\log 1036 = 3{\cdot}0128$$
$$\phantom{\log 1036 = 3{\cdot}01}26$$
$$= 3{\cdot}0154$$
$$\log 1076 = 3{\cdot}0294$$
$$\phantom{\log 1076 = 3{\cdot}02}24$$
$$= 3{\cdot}0318.$$

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	[illegible]	[illegible]

66. To find the Number, given the logarithm of the number.

The *characteristic* of the logarithm, increased by 1 if positive, gives the number of figures to the left of the decimal point in the Number required, and diminished by 1 if negative, gives the number of ciphers to the right of the decimal point in the number required.

From the *mantissa* we find the significant figures of the required number from the *Antilogarithm Table.* We look for the *first two* figures of the mantissa in the first column, and passing along the row containing these, take the number in that particular column headed by the *third* figure of the mantissa; to this number is added the number in that particular difference column headed by the *fourth* figure of the mantissa.

Ex. 1. Find the number whose logarithm is 2·3175, *i.e.* find x, when $\log x = 2{\cdot}3175$.

Since the characteristic is 2, there will be 3 figures to the left of the decimal point.

We look out the row containing ·31 in the first column and in this row select the number in the column headed by the third figure 7 of the mantissa; we thus obtain 2075. In this same row the number in the difference column headed by the fourth figure 5 is 2; adding this to the previous result we obtain 2077.

$\therefore$ number is 207·7.

Ex. 2. Find the number whose logarithm is 7·4235.

There will be 8 figures to the left of the decimal point in the required number.

We look out the row containing ·42 in the first column and select in this row the number in the column headed by 3, this gives 2649. In the same row, the number in the difference column headed by 5 is 3; adding to the previous result we obtain 2652.

$$\therefore \text{ number} = 26520000.$$

When the number ends with ciphers, it is better to write it as a multiple of a power of 10.

Thus $$\text{number} = 2{\cdot}652 \times 10^7.$$

[Observe that the index 7 is the same as the characteristic.]

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6

67. Instead of using the Antilogarithm Table, the ordinary Logarithm Table may be utilised for finding the number whose logarithm is given. Thus in Ex. 2 we look for the row containing the number nearest to 4235 and find it to be the row starting with **26** and containing 4232, in a column headed by **5.** We now have to account for a difference of 3 in the last figure and find it in a difference column headed by **2.** Thus the significant figures we require are 2652 and since the characteristic is 7, the number is $2{\cdot}652 \times 10^7$.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15

N.B. If the logarithm had been 7·4236, it will be noticed that no difference column contains 4; in such a case we select from the difference columns the number nearest to that required, and here again 4 being equidistant from the contiguous numbers 3 and 5, we choose the larger 5.

Ex. 1. Find the value of

$$\frac{37{\cdot}21 \times 82{\cdot}33}{{\cdot}4729}.$$

Let $\quad x = \text{given expression.}$

$$\therefore \log x = \log 37{\cdot}21 + \log 82{\cdot}33 - \log {\cdot}4729,$$

$$\begin{array}{ll} \log 37{\cdot}21 = & 1{\cdot}5706 \\ \log 82{\cdot}33 & \underline{1{\cdot}9156} \\ & 3{\cdot}4862 \\ \log {\cdot}4729 & \bar{1}{\cdot}6747, \end{array}$$

$$\therefore \log x = 3{\cdot}8115,$$

$$\therefore x = 6479.$$

Ex. 2. Find the value of $(5{\cdot}726)^8$.

$$\log x = 8 \log 5{\cdot}726$$
$$= 8 \times {\cdot}7579$$
$$= 6{\cdot}0632,$$
$$\therefore x\,(= 1157000) = 1{\cdot}157 \times 10^6.$$

Ex. 3. Find the value of $({\cdot}02357)^{\frac{1}{6}}$.

$$\log x = \tfrac{1}{6} \log {\cdot}02357$$
$$= \tfrac{1}{6} \times \bar{2}{\cdot}3724$$
$$= \bar{1}{\cdot}7287 \text{ (correct to 4 places),}$$
$$\therefore x = {\cdot}5355.$$

Ex. 4. The volume of a hollow circular cylinder of external radius R, internal radius r and length l is $\pi(R^2 - r^2)\,l$.

Find the volume when

$$R = 78{\cdot}42 \text{ metres,} \qquad r = 39{\cdot}25 \text{ m.,}$$
$$l = 127{\cdot}32 \text{ metres, and } \pi = 3{\cdot}142.$$

$$V = 3{\cdot}142 \times (78{\cdot}42 - 39{\cdot}25)(78{\cdot}42 + 39{\cdot}25) \times 127{\cdot}3 \text{ (approx.)}$$
$$= 3{\cdot}142 \times 39{\cdot}17 \times 117{\cdot}67 \times 127{\cdot}3 \text{ cu. m.}$$

$$\begin{array}{lll} \log & 3{\cdot}142 = & {\cdot}4972, \\ \log & 39{\cdot}17 & 1{\cdot}5930, \\ \log & 117{\cdot}7 & 2{\cdot}0708, \\ \log & 127{\cdot}3 & \underline{2{\cdot}1048,} \end{array}$$

$$\therefore \log V = 6{\cdot}2658.$$

$$\therefore V\,(= 1844000) = 1{\cdot}844 \times 10^6 \text{ cu. m.}$$

Ex. 5. The periodic time T (in seconds) of a mass m suspended at one end of a spiral spring, the other end being fixed, is given by $T = 2\pi\sqrt{\dfrac{m}{F}}$, where F is the force producing unit displacement.

If $\pi = 3{\cdot}1416$, $F = 400$ poundals and $m = 12{\cdot}9$ lbs., find T.

$$\log T = \log 6{\cdot}2832 + \tfrac{1}{2}(\log 12{\cdot}9 - \log 400),$$

$$\begin{aligned}
\log 6{\cdot}283 &= {\cdot}7982 \\
\tfrac{1}{2}\log 12{\cdot}9 &= \underline{{\cdot}5553} \\
& 1{\cdot}3535. \\
\tfrac{1}{2}\log 400 &= \underline{1{\cdot}3011} \quad \text{(correct to 4 places).} \\
\therefore \log T &= {\cdot}0524, \\
\therefore T &= 1{\cdot}128 \text{ sec.}
\end{aligned}$$

Ex. 6. Solve the equation $5^x \cdot 7^{x+1} = 13^{2x+1}$.

Taking logarithms,

$$x\log 5 + (x+1)\log 7 = (2x+1)\log 13.$$

$$\therefore (x \times {\cdot}6990) + (x+1)\,{\cdot}8451 = (2x+1)\,1{\cdot}1139.$$

$$\therefore {\cdot}6837x = -{\cdot}2688.$$

$$\therefore x = -{\cdot}39 \text{ (correct to 2 places).}$$

EXAMPLES XV.

Find the value of

1. $7{\cdot}203 \times 823{\cdot}1$.
2. $5972{\cdot}6 \times 81{\cdot}32 \times 57{\cdot}67$.
3. $8275{\cdot}7 \times 5297{\cdot}6 \times {\cdot}00345$.
4. $\dfrac{815{\cdot}9 \times {\cdot}00326}{{\cdot}7185}$.
5. $\dfrac{598{\cdot}5 \times {\cdot}07281 \times 5{\cdot}279}{82{\cdot}34}$.
6. $(523{\cdot}7)^{\frac{2}{7}}$.

7. $\sqrt[3]{7892} \times \sqrt[5]{87 \cdot 45}$.

8. $\sqrt[5]{72 \cdot 96} \times \sqrt{8753 \cdot 2} \times \sqrt{724 \cdot 8}$.

9. $(823 \cdot 9)^{\frac{1}{3}} \times (72 \cdot 54)^{\frac{2}{5}} \times (3 \cdot 146)^{\frac{1}{7}}$.

10. $\dfrac{(32 \cdot 7)^3 \times (82 \cdot 75)^{\frac{1}{3}} \times (97 \cdot 62)^{\frac{3}{5}}}{(87 \cdot 62)^{\frac{1}{5}}}$.

11. If the time of oscillation (in seconds) of a pendulum is given by $T = 2\pi\sqrt{\dfrac{l}{g}}$, find the time when $\pi = 3 \cdot 1416$, $l = 565$ cms., and $g = 981$.

12. Find the volume of a sphere of radius 678 centimetres from the formula $\frac{4}{3}\pi r^3$, where $\pi = 3 \cdot 1416$.

13. With certain data it is found that if l is the height of a pine tree in centimetres,

$$l^3 < \frac{7 \cdot 84 \times 10^{11} \times 15^2}{\cdot 6 \times 981 \times 16}.$$

What is the maximum value of l obtained from this?

14. If a certain spring is loaded with a kilogram the depression is given by $d = \dfrac{600 \times 981 \times 10^3 \times (1 \cdot 5)^4}{\pi \times 8 \times 10^{11} \times 10^{-4}}$ centimetres.

Calculate the value of d. $(\pi = 3 \cdot 1416.)$

15. The loss in Kinetic Energy after impact of 2 balls of masses M and m moving with velocities v and u before impact, is given by $\frac{1}{2}(1 - e^2)\dfrac{Mm}{M + m}(v - u)^2$, where e is the coefficient of elasticity. Find the value of this expression when $M = 257 \cdot 5$ grams, $m = 201 \cdot 6$ grams, $v = 11$ centimetres per second, $u = 9$ centimetres per second and $e = \cdot 66$.

16. Find in cu. decimetres the volume of a cylinder of height $27 \cdot 27$ dm. and the radius of whose circular base is $5 \cdot 37$ dm., given that Volume $= \pi r^2 h$. $(\pi = 3 \cdot 1416.)$

17. The ratio of the work done to the heat generated in Joule's experiment is $\dfrac{4 n\pi a W}{(M + m)\,t}$. Find the value of this ratio when $M + m = 84280$ grains, $2W = 18229$ grains, $2\pi a = 2 \cdot 774$ ft., $n = 4870$, $t = 3^\circ \cdot 768$.

18. The weight of water vapour obtained in a certain experiment was $\dfrac{10 \times 12{\cdot}7 \times 273 \times {\cdot}806}{760 \times 288}$ grams. Calculate the value of this fraction.

19. From observations on the boiling point at two stations the difference in level is found to be $\dfrac{13{\cdot}59 \times 11{\cdot}82 \times 760 \times 283}{{\cdot}001293 \times 692{\cdot}8 \times 273}$ cms. Calculate the value of this difference in level.

20. The strength of a magnetic field is found from the formula $H = \pi n \sqrt{\left(\dfrac{2K}{r^3 \tan\theta}\right)}$. Calculate H when $\pi = 3{\cdot}1416$, $n = {\cdot}144$, $K = 379{\cdot}9$, $r = 40$, $\tan\theta = {\cdot}0787$.

Solve the equations (giving x correct to 2 places of decimals):

21. $3^{2x} . 5^{x} = 7^{3x+1}$.

22. $5^{3x+2} . 7^{2x+1} = 11^{x}$.

23. $3^{2x+1} . 11^{x} = 13^{x+5}$.

24. $2^{2x+1} . 7^{x+3} = 17^{x+5}$.

25. The parallax (P) of Castor is $0{\cdot}2''$; find its distance in miles from the formula, distance $= 206265 \dfrac{r}{P''}$ miles, where $r =$ radius of earth's orbit $= 93{,}000{,}000$ miles.

26. From the same formula, find the distance of α Centauri, whose parallax is $0{\cdot}750''$.

27. The time taken by light to travel from a star to the earth is $\dfrac{K}{2\pi P}$ years, where $K = 20{\cdot}49$ and P is the star's parallax in seconds. Find this value in the case of α Centauri, the nearest fixed star, whose parallax is $0{\cdot}750''$. $(\pi = 3{\cdot}1416.)$

28. Find from the tables the logarithms of 400, 401, 402. Represent the change in the logarithms on squared paper and thence deduce the values of $\log 400{\cdot}4$ and $\log 401{\cdot}7$.

29. Find from the tables the logarithms of 331, 332, 333 and show the increase in the logarithms by a graph, from which deduce the values of $\log 331{\cdot}2$ and $\log 332{\cdot}8$.

TABLES OF LOGARITHMIC SINES, COSINES, ETC.

68. Since the sine and cosine are never greater than 1, their logarithms are always negative; it is thus found convenient to add 10 to the logarithms of the Trigonometrical Functions, and they are then known as Tabular Logarithmic Sines, Cosines, etc., or shortly Logarithmic Sines, Cosines, etc. The notation is $L \sin \mathrm{A}$

and obviously $\quad L \sin \mathrm{A} = 10 + \log \sin \mathrm{A}$,

$$\text{e.g. } \log \sin 60^\circ = \log \frac{\sqrt{3}}{2} = \tfrac{1}{2}\log 3 - \log 2 = \cdot 2385 - \cdot 3010$$

$$= \bar{1}\cdot 9375,$$

$$\therefore L \sin 60^\circ = 9\cdot 9375.$$

69. In Tables of 4 figure logarithms, the logarithmic sines, cosines, tangents, cosecants, secants and cotangents are given for all angles between 0° and 90° at intervals of 6 minutes; difference columns are provided for angles of 1, 2, 3, 4, 5 minutes. The numbers found in the difference columns are *added* in the case of the sine, tangent, and secant, since these functions increase from 0° to 90°, and *subtracted* in the case of the *cosine*, *cotangent* and *cosecant*, since these functions diminish as the angle increases from 0° to 90°.

Ex. 1. Find $L \sin 54^\circ 34'$.

Turning to the page of logarithmic sines, we look in the first column for 54° and along the row containing 54° to the number in the column headed by 30′ (the number next below that required), this gives 9·9107. We now have to find the difference for 4′, and looking in the difference column for 4′ and in the same row as before, we obtain the number 4.

Thus $\quad L \sin 54^\circ 30' = 9\cdot 9107,$

$\quad\quad$ *diff. for* $4' = \quad 4,$

$\therefore L \sin 54^\circ 34' = 9\cdot 9111$ (adding for the sine).

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
54	9·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5

x. 2. Find $L \cos 12° 44'$.

; in the last example

$$L \cos 12° 42' = 9{\cdot}9892,$$
$$\textit{diff. for } 2' = \quad 1,$$
$$\therefore\ L \cos 12° 44' = 9{\cdot}9891$$

(subtracting for the cosine).

o student should check his work, thus :

ow $\cos 12° 42' > \cos 12° 44'$,

$$\therefore\ L \cos 12° 42' > L \cos 12° 44'.$$

' in looking out any value we find a bar placed over the 'igure, this means that the integer in the next row is to .ed. Thus

$$L \sec 84° 24' = 11{\cdot}0106 \text{ and not } 10{\cdot}0106.$$

′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
808	9880	9951	$\overline{0}030$	$\overline{0}108$	$\overline{0}184$	$\overline{0}261$	$\overline{0}346$	$\overline{0}427$	$\overline{0}511$	13	26	39	53	66
597	0685	0771	0865	0958	1054	1151	1251	1353	1457	16	32	48	64	81

'hese same tables are used to find the value of the angle, 1 the logarithm.

Ex. 3. Find x, when $L \sec x = 10{\cdot}3425$.

'urning to the page of logarithmic secants, we look out the oer nearest to 10·3425 and smaller than it; we find that

$$L \sec 62° 54' = 10{\cdot}3415.$$

'hen in the same line, we look for a difference of

$$10\ (= 3425 - 3415),$$

find that it corresponds to 4′.

$$\therefore\ L \sec 62° 58' = 10{\cdot}3425,$$
$$x = 62° 58'.$$

If the difference columns do not contain the necessary ıber, we take the nearest; and if the number is midway veen two given in the difference column, we take the er.

Two examples will be considered to illustrate this point.

Ex. 4. Given that L cosec $x = 10{\cdot}9217$, find x.

Using the Tables as in the last example, we find that

$$L \text{ cosec } 6^\circ\, 48' = 10{\cdot}9266$$

(taking the number nearest to 9217 and greater than it, since we have to subtract this time).

We now have to account for a difference of 49, and find that a difference of 44 corresponds to 4′ and a difference of 56 to 5′; and 49 being nearer to 44 than to 56′, we take 4′.

$$\therefore\ x = 6^\circ\, 52'.$$

Ex. 5. Given that L cosec $x = 10{\cdot}6884$, find x.

$$L \text{ cosec } 11^\circ\, 48' = 10{\cdot}6893.$$

The difference for 1′ is 6 and for 2′ is 12, whereas we have to account for a difference of 9, which is midway between 6 and 12; we select the larger angle 2′ (except as in Art. 86).

$$\therefore\ x = 11^\circ\, 50'.$$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
6	10·9808	9736	9666	9597	9528	9461	9395	9330	9266	9203	11	22	33	44	56
11	10·7194	7155	7117	7079	7041	7003	6966	6930	6893	6857	6	12	19	25	31

Ex. 6. Find the value of sec $108^\circ \times$ tan $25^\circ\, 13'$.

Exp. $= -\sec 72^\circ \times \tan 25^\circ\, 13' = -x$ (suppose).

$$\therefore\ \log x = L \sec 72^\circ + L \tan 25^\circ\, 13' - 20$$

$$= 10{\cdot}5100$$

$$+\,9{\cdot}6729 - 20$$

$$= {\cdot}1829.$$

$\therefore$ Expression $= -x = -1{\cdot}524.$

Ex. 7. The deviation (D″) of the vertical due to centrifugal force in latitude l is $\dfrac{180 \times 60 \times 60}{289 \times 2\pi} \sin 2l$.

Find D when $l = 55^\circ$ and $\pi = 3{\cdot}142$.

$$\log D = \log 324000 + L \sin 110^\circ - \log 289 - \log 3{\cdot}142 - 10$$

$$\begin{array}{rl} = 5{\cdot}5105 - & 2{\cdot}4609 \\ 9{\cdot}9730 & 10{\cdot}4972 \\ \hline 15{\cdot}4835 & 12{\cdot}9581 \end{array}$$

$= 2{\cdot}5254.$

$$\therefore D = 335{\cdot}3''.$$

70. Since $\cos\alpha = \sin(90^\circ - \alpha)$

$$\cot\alpha = \tan(90^\circ - \alpha)$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\sin(90^\circ - \alpha)}$$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha},$$

tables of logarithmic sines and tangents would be sufficient to work out all examples.

EXAMPLES XVI.

Using Logarithmic Tables, find the values of

1. $\sin 37^\circ 24' \times \cos 72^\circ 15'$.
2. $\sin 26^\circ 32' \times \cot 41^\circ 17'$.
3. $\tan 37^\circ 33' \times \operatorname{cosec} 22^\circ 18'$.
4. $\sec 53^\circ 22' \times \operatorname{cosec} 22^\circ 27'$.
5. $\cot 125^\circ 47' \times \cos 172^\circ 15'$.
6. $\sin 153^\circ 44' \times \tan 73^\circ 27'$.

7. $\tan 127° 31' \times \cot 136° 11'$.

8. $\operatorname{cosec} 143° 22' \times \cot 157° 3'$.

9. $\cos 152° 13' \times \sec 36° 2'$.

10. $\sin 37° 15'/\cos 47° 13'$.

11. $\tan 57° 32'/\cot 47° 3'$.

12. $\sec 22° 13'/\tan 51° 41'$.

13. At the equinox, in latitude l, the time taken for the sun to rise is $\frac{1}{15}D'' \sec l$ seconds. Find the value of this in seconds when D = sun's diameter = 32′ and $l = 52° 31'$.

14. The coefficient of diurnal aberration is $\dfrac{15a \cos l}{V}$ where a = radius of earth = 3960 miles, V = velocity of light = 186000 miles per sec., and l = observer's latitude = 25° 31′. Find the value.

15. With a conical pendulum of length l feet, making n revolutions per second, the angle of inclination of the string to the vertical is θ, where $\cos \theta = \dfrac{g}{4n^2\pi^2 l}$; find θ, when $g = 32$, $n = \cdot 7$, $\pi = 3\cdot 142$, $l = 12\cdot 4$.

16. In latitude l, the Earth's rotation diminishes the weight of a body by $\frac{1}{289}\cos^2 l$ of itself. Calculate this fraction when $l = 55°$.

17. At either equinox, in latitude l, a mountain whose height is $\dfrac{1}{n}$ of earth's radius, catches the sun's rays in the morning

$$\frac{12}{\pi \cos l}\sqrt{\frac{2}{n}} \text{ hours before he rises on the plain at the base.}$$

Calculate this time when

$$\pi = 3\cdot 142,\ l = 42° 32',\ \text{mountain} = 14000 \text{ ft.},$$

and earth's radius = 4000 miles.

18. If a body is projected with a ve. roportional to second up an inclined plane, angle β, at a. horizontal, then the greatest distance reached l the plane is $\dfrac{u^2 \sin(\alpha - \beta)}{2g \cos \beta}$ feet. Calculate this $u = 59{\cdot}7$ feet per second, $\alpha = 75°$, $\beta = 32° 13'$, $2g =$

19. The range up a similar plane is $\dfrac{2u^2 \cos \alpha .}{g \cos^2 \beta}$ feet. Calculate the range when $u = 47{\cdot}5$, $\alpha = 58°$, $\beta = 33° 15'$, $g = 32$.

20. The angular elevation of a fort on a hill h feet high is β; in order to hit it, the initial velocity must be not less than $\sqrt{gh(1 + \operatorname{cosec} \beta)}$ feet per second. Calculate this value when $g = 32$, $h = 527$, $\beta = 12° 23'$.

18. If a body is projected with a velocity of u feet per second up an inclined plane, $\angle\beta$, at an $\angle\alpha$ to the horizontal, then the greatest distance reached $\underline{\text{l}}$ to the plane is $\dfrac{u^2 \sin(\alpha - \beta)}{2g \cos\beta}$ feet.

Calculate this distance when $u = 59{\cdot}7$ feet per second, $\alpha = 75°$, $\beta = 32° 13'$, $2g = 64$.

19. The range up a similar plane is $\dfrac{2u^2 \cos\alpha \sin(\alpha - \beta)}{g \cos^2\beta}$ feet. Calculate the range when $u = 47{\cdot}5$, $\alpha = 58°$, $\beta = 33° 15'$, $g = 32$.

Miscellaneous Examples on Chapter VII start on page 218, Test Paper XXIII.

7. tan 127°

8. cosec

9. cos

10.

1

CHAPTER VIII.

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

71. THE following important formulae are proved in this chapter.

(1) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R,$

(2) $a = b\cos C + c\cos B,$

(3) $c^2 = a^2 + b^2 - 2ab\cos C,$

(4) $\sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)},$

(5) $\Delta = \frac{1}{2}ab\sin C$

$= \sqrt{s(s-a)(s-b)(s-c)},$

(6) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}},$

(7) $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}},$

(8) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}},$

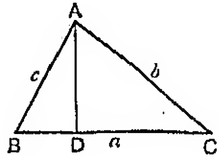
(9) $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}.$

It has formerly been the custom to prove formulae 6–9 as in Chapter XIII. If thought advisable the student may (i) defer the *proofs* of these formulae, using the *results* for the examples on Solution of Triangles, or (ii) omit these formulae and take Chapters XI, XII, XIII before Chapter IX.

72. The sides of a triangle are proportional to the sines of the opposite angles, *i.e.*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

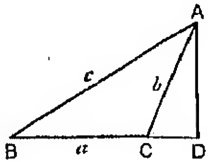
1*st Method.*



Draw a perp. AD from A to BC or BC produced.

Then in both cases

$$\frac{AD}{AB} = \sin B, \quad \therefore \quad AD = c \sin B.$$



In Fig. 1 $\quad \dfrac{AD}{AC} = \sin C.$

In Fig. 2 $\quad \dfrac{AD}{AC} = \sin(180^\circ - C) = \sin C.$

$\therefore$ in both cases, $\quad AD = b \sin C.$

Equating these values of AD,

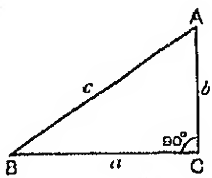
$$b \sin C = c \sin B$$

or

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}.$

If $C = 90^\circ$, then $\sin C = 1$ and the relation becomes



$$\frac{a}{\sin A} = \frac{b}{\sin B} = c,$$

or the well-known formulae

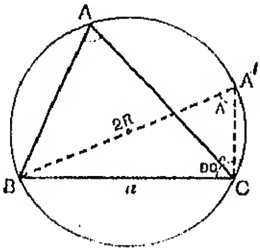
$$\sin A = \frac{a}{c},$$

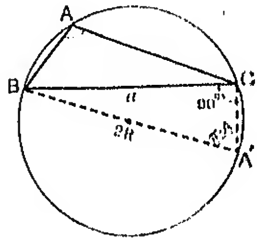
$$\sin B = \frac{b}{c}.$$

73. *2nd Method.*

Draw the circumscribing circle and let BA′ be the diameter through B.

Since the angle in a semi-circle is a right angle.





In Fig. 1, $$\frac{BC}{BA'}=\sin A.$$

In Fig. 2, $$\frac{BC}{BA'}=\sin(\pi-A)$$
$$=\sin A,$$

$$\therefore \frac{a}{2R}=\sin A,$$

or $$\frac{a}{\sin A}=2R,$$

where R is the radius of the circum-circle.

Similarly, $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R.$$

Since $$a=2R\sin A,$$

$\therefore$ **any chord of a circle = 2R × sine of the angle it subtends at the circumference.**

It follows that

Any two chords of a circle bear to one another the same ratio as the sines of the angles which they subtend at the circumference.

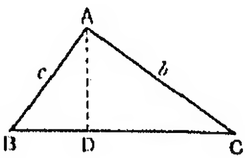
74. To prove

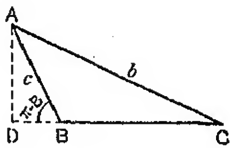
$$a = b\cos C + c\cos B$$

$$b = a\cos C + c\cos A$$

$$c = a\cos B + b\cos A.$$

Draw AD perpendicular to BC.





In Fig. 1, $\quad BC = BD + DC,$

$$\therefore\ a = c\cos B + b\cos C.$$

In Fig. 2, $\quad BC = CD - DB,$

$$\therefore\ a = b\cos C - c\cos(\pi - B)$$

$$a = b\cos C + c\cos B.$$

The other results may be proved by drawing perpendiculars to the sides AC and AB, or we may say they follow directly by symmetry.

This proposition may be stated in words:

Any side of a triangle

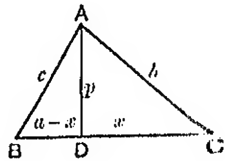
= the sum of the projections of the other two sides on it.

75. *To find the cosine of an angle in terms of the sides of the triangle.*

Draw a perp. from A to BC or BC produced.

1st Method.

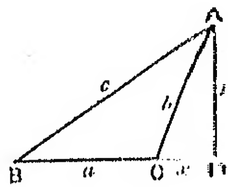
Let $AD = p$, $CD = x$.



If angle C is acute,

$$\begin{aligned} c^2 &= p^2 + (a - x)^2 \\ &= (p^2 + x^2) + a^2 - 2ax \\ &= b^2 + a^2 - 2ax \\ &= b^2 + a^2 - 2ab \cos C. \end{aligned}$$

If angle C is obtuse,



$$\begin{aligned} c^2 &= p^2 + (a + x)^2 \\ &= (p^2 + x^2) + a^2 + 2ax \\ &= b^2 + a^2 + 2ab \cos (180^\circ - C) \\ &= b^2 + a^2 - 2ab \cos C. \end{aligned}$$

$\therefore$ in both cases,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$$2ab \cos C = a^2 + b^2 - c^2,$$

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \text{..................(i).}$$

Similarly $$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \quad \text{..................(ii).}$$

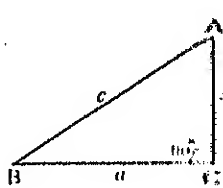
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{..................(iii).}$$

If $C = 90^\circ$, then $\cos C = 0$,

$\therefore$ from (i) $\quad c^2 = a^2 + b^2,$

from (ii) $\quad \cos B = \dfrac{2a^2}{2ca} = \dfrac{a}{c},$

from (iii) $\quad \cos A = \dfrac{2b^2}{2bc} = \dfrac{b}{c}.$



$$
\begin{aligned}
AB^2 &= AD^2 + BD^2 \\
&\left.\begin{aligned} &= AD^2 + (BC - CD)^2 \\ &= b^2 \sin^2 C + (BC - b\cos C)^2 \end{aligned}\right\} \text{ if C is acute} \\
&\left.\begin{aligned} &= AD^2 + (BC + CD)^2 \\ &= b^2 \sin^2 (\pi - C) + \{BC + b\cos(\pi - C)\}^2 \\ &= b^2 \sin^2 C + (BC - b\cos C)^2 \end{aligned}\right\} \text{ if C is obtuse.}
\end{aligned}
$$

$\therefore$ $AB^2 = b^2 \sin^2 C + (BC - b\cos C)^2$ in both cases,

$$
\begin{aligned}
\therefore c^2 &= b^2 \sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C \\
&= b^2 + a^2 - 2ab\cos C.
\end{aligned}
$$

77. *To find the sine of an angle in terms of the sides of a triangle.*

By previous article

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$
\begin{aligned}
\therefore \sin^2 A &= 1 - \cos^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 \\
&= \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right) \\
&= \frac{(b+c)^2 - a^2}{2bc} \cdot \frac{a^2 - (b-c)^2}{2bc} \\
&= \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4b^2c^2} \\
&= \frac{4s(s-a)(s-b)(s-c)}{b^2c^2},
\end{aligned}
$$

where $2s = a + b + c$,

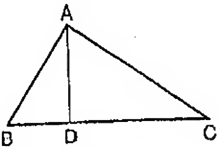
$$\therefore \sin A = \pm \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

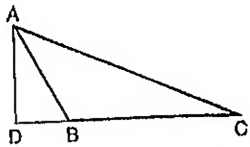
and the $-$ sign may be neglected because A is less than 180°, and therefore sin A is positive.

Similarly $\quad \sin B = \dfrac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)},$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

78. *To find the area* (Δ) *of a triangle.*





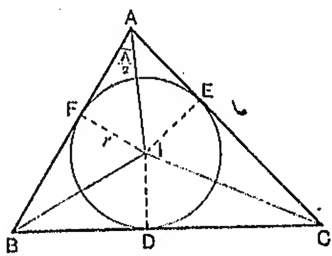
By geometry

$$\begin{aligned} \text{area} &= \tfrac{1}{2} \,.\, \text{AD} \,.\, \text{BC} \\ &= \tfrac{1}{2} \,.\, c \sin \text{B} \,.\, a \\ &= \tfrac{1}{2}\, ca \sin \text{B} \\ &= \tfrac{1}{2} \,.\, ca \,.\, \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}, \end{aligned}$$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

79. *To find the radius* (r) *of the circle inscribed in a triangle*, and hence (Arts. 81, 82, 83) the value of the sine, cosine and tangent of

$$\frac{\text{A}}{2}, \frac{\text{B}}{2}, \frac{\text{C}}{2}.$$



By Geometry the lines bisecting the angles of the triangle meet at the centre of the inscribed circle.

Draw these bisectors meeting at I, and draw ID, IE, IF perpendicular to the sides,

then $$\text{ID} = \text{IE} = \text{IF} = r.$$

Now $$\begin{aligned} \Delta &= \text{area of BIC} + \text{AIC} + \text{AIB} \\ &= \tfrac{1}{2} \,.\, r \,.\, \text{BC} + \tfrac{1}{2} \,.\, r \,.\, \text{CA} + \tfrac{1}{2} \,.\, r \,.\, \text{AB} \\ &= \tfrac{1}{2} \,.\, r(a+b+c) \\ &= rs. \end{aligned}$$

$$\therefore r = \frac{\Delta}{s} = \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}.$$

80. To prove

$$r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}.$$

The tangents from any point to a circle are equal.

$$\therefore\ AF = AE;\ BF = BD;\ CE = CD.$$

Now $$AF + AE + BF + BD + CE + CD = 2s,$$

$$\therefore\ 2AF + 2BD + 2CD = 2s,$$

$$\therefore\ AF + BC = s,$$

$$\therefore\ AF = s - a.$$

Similarly $$BD = s - b;\ CD = s - c.$$

Now $$\frac{r}{AF} = \tan\frac{A}{2},$$

$$\therefore\ \frac{r}{s-a} = \tan\frac{A}{2},$$

$$\therefore\ r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}.$$

* **81.** To prove $$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\frac{\Delta}{s} = r = (s-a)\tan\frac{A}{2},$$

$$\therefore\ \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = (s-a)\tan\frac{A}{2},$$

$$\therefore\ \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly $$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

* For alternative proofs, see pp. 117*a*, 117*b*.

***82.** To prove $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$.

$$\sec^2\frac{A}{2}=1+\tan^2\frac{A}{2}$$
$$=1+\frac{(s-b)(s-c)}{s(s-a)}$$
$$=\frac{2s^2-s(a+b+c)+bc}{s(s-a)}$$
$$=\frac{2s^2-s\,.\,2s+bc}{s(s-a)}$$
$$=\frac{bc}{s(s-a)};$$
$$\therefore \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

the value $\cos\frac{A}{2}=-\sqrt{\frac{s(s-a)}{bc}}$ is discarded

because $\frac{A}{2}<90^\circ$ and $\therefore$ $\cos\frac{A}{2}$ is positive.

Similarly $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$,

$$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

***83.** To prove

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$\sin\frac{A}{2}=\tan\frac{A}{2}\cos\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

(Arts. 81, 82).

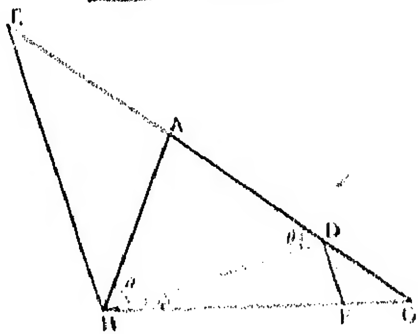
Similarly $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$,

$$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

* For alternative proofs, see pp. 117a, 117b.

To prove the formula

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$



ce CA to E and make

$$AE = AD = AB\ (=c);$$

DB and draw DF parallel to EB.

$$A\hat{B}D = A\hat{D}B = \theta,$$

$$D\hat{B}C = \phi;$$

$$\theta + \phi = A\hat{B}D + D\hat{B}C = B,$$

$$\theta - \phi = A\hat{D}B - D\hat{B}C = D\hat{C}B = C.$$

$$\therefore\ \theta = \tfrac{1}{2}(B + C) = 90^\circ - \frac{A}{2},$$

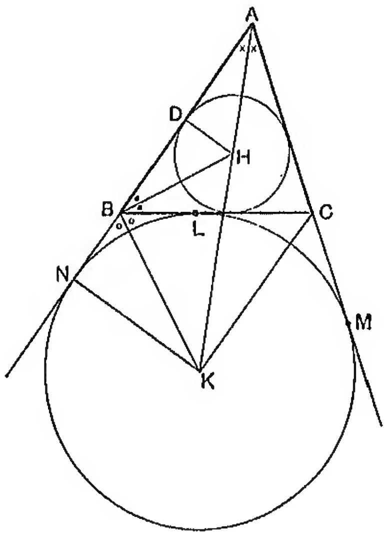
$$\phi = \tfrac{1}{2}(B - C).$$

$$\frac{b-c}{b+c} = \frac{DC}{EC} = \frac{DF}{EB} = \frac{DF}{BD}\Big/\frac{EB}{BD}$$

$$= \frac{\tan\phi}{\tan\theta}, \quad \text{since } B\hat{D}F = D\hat{B}E = 90^\circ,$$

$$= \frac{\tan\frac{1}{2}(B-C)}{\cot\frac{A}{2}}.$$

Lemma. Draw the inscribed and one of the escribed circles of the triangle ABC; let H and K be the centres.



Let BL = BN = x; CL = CM = $a - x$.

$$\therefore \text{AN} = \text{AB} + \text{BN} = \text{AB} + \text{BL} = c + x,$$

$$\text{AM} = \text{AC} + \text{CM} = \text{AC} + \text{CL} = b + a - x.$$

$$\therefore c + x = b + a - x.$$

$$\therefore \text{BN} = x = \frac{a + b - c}{2} = s - c.$$

Also $\text{AN} = \text{AM} = \frac{1}{2}(\text{AN} + \text{AM}) = \frac{a + b + c}{2} = s.$

(See also Art. 145.)

***81.**
$$\tan^2\frac{A}{2}=\frac{KN}{NA}\cdot\frac{HD}{DA}.$$

Now since BK and BH are the external and internal bisectors of the angle ABC and are consequently at right angles,

$$\therefore B\hat{K}N = 90^\circ - N\hat{B}K = D\hat{B}H.$$

Thus the right-angled triangles KNB and BDH are similar,

$$\therefore KN \,.\, HD = BD \,.\, BN.$$

$$\therefore \tan^2\frac{A}{2}=\frac{BD\,.\,BN}{NA\,.\,DA}=\frac{(s-b)(s-c)}{s(s-a)}. \qquad \text{(Art. 80.)}$$

***82.**
$$\cos^2\frac{A}{2}=\frac{AN}{AK}\cdot\frac{AD}{AH}.$$

Now in the triangles BHK and AKC,

$$B\hat{H}K = B\hat{A}H + H\hat{B}A = \tfrac{1}{2}(A+B),$$

$$K\hat{C}M = \tfrac{1}{2}B\hat{C}M = \tfrac{1}{2}(A+B).$$

$$\therefore A\hat{H}B = 180^\circ - B\hat{H}K = 180^\circ - K\hat{C}M = A\hat{C}K,$$

also $\quad B\hat{A}H = \dfrac{A}{2} = K\hat{A}C.$

$\therefore$ the triangles BHK and AKC are similar,

and $$AK \,.\, AH = AC \,.\, AB;$$

$$\therefore \cos^2\frac{A}{2}=\frac{AN\,.\,AD}{AC\,.\,AB}=\frac{s(s-a)}{bc}. \qquad \text{(Art. 80.)}$$

***83.**
$$\sin^2\frac{A}{2}=\frac{NK}{KA}\cdot\frac{DH}{HA}=\frac{BD\,.\,BN}{AC\,.\,AB},$$

since the triangles KNB and BDH are similar, also the triangles BHK and AKC are similar;

$$\therefore \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}.$$

Ex. 1. Find the area of the triangle when

$$a = 18{\cdot}2 \text{ cm.}, \quad b = 16{\cdot}4 \text{ cm.}, \quad c = 14{\cdot}6 \text{ cm.}$$

$$\begin{array}{ll} a = 18{\cdot}2^* & \therefore\ s = 24{\cdot}6^*, \\ b = 16{\cdot}4 & s - a = \ \ 6{\cdot}4, \\ c = \underline{14{\cdot}6} & s - b = \ \ 8{\cdot}2, \\ \quad\ 49{\cdot}2 & s - c = 10\ . \end{array}$$

$$\therefore\ \Delta = \sqrt{24{\cdot}6 \times 6{\cdot}4 \times 8{\cdot}2 \times 10},$$

$$\log \Delta = \tfrac{1}{2}(\log 24{\cdot}6 + \log 6{\cdot}4 + \log 82),$$

$$\begin{array}{lr} \log 24{\cdot}6 = & 1{\cdot}3909 \\ \log\ \ 6{\cdot}4 = & {\cdot}8062 \\ \log 82 \ \ = & 1{\cdot}9138 \\ & 2\,\underline{|\,4{\cdot}1109} \\ \therefore\ \log \Delta = & 2{\cdot}0555 \end{array}$$

$$\therefore\ \Delta = 113{\cdot}6 \text{ sq. cm.}$$

Ex. 2. If $a = 10$, $b = 12$ and $C = 35°$, find c.

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$$\begin{aligned} \therefore\ c^2 &= 100 + 144 - (240 \times {\cdot}8192) \\ &= 244 - 196{\cdot}608 \\ &= 47{\cdot}392, \end{aligned}$$

$$\therefore\ c = 6{\cdot}88 \text{ (approx.)}.$$

Ex. 3. In any triangle prove that

$$\cos B\,(b - a\cos A) = \cos C\,(c - b\cos A).$$

$$\begin{aligned} \cos B\,(b - c\cos A) &= \cos B\,(a\cos C + c\cos A - c\cos A) \\ &= a\cos B\cos C \\ &= \cos C\,(c - b\cos A). \end{aligned}$$

* Note that the sums of the numbers in these two columns are the same.

EXAMPLES XVII.

(On the use of formulae 1—5, Art. 71.)

Find the area of the triangle, given

1. $a=17 \cdot 2$ cm., $b=15 \cdot 3$ cm., $c=14 \cdot 9$ cm.

2. $a=25$ cm., $b=26$ cm., $c=18 \cdot 5$ cm.

3. $a=18 \cdot 24$ cm., $b=19 \cdot 36$ cm., $c=14 \cdot 22$ cm.

4. If $a=17$, $b=11$ and $C=42^\circ$, find c.

5. $b=16$, $c=14$ and $A=72^\circ$, find a.

6. $c=18$, $a=5$ and $B=34^\circ$, find b.

7. Find $\sin A$, if $a=14 \cdot 2$, $b=12 \cdot 8$, $c=10 \cdot 4$.

8. $\sin B$, if $a=18 \cdot 2$, $b=10 \cdot 4$, $c=16 \cdot 8$.

In any triangle, prove that

9. $b^2 - c^2 = a(b \cos C - c \cos B)$.

10. $\dfrac{b - a \cos C}{c - a \cos B} = \dfrac{\sin C}{\sin B}$.

11. $(a+b)(1 - \cos C) = c(\cos A + \cos B)$.

12. $4\Delta \cot A = b^2 + c^2 - a^2$.

13. $a(\cos B - \sin B) + b(\cos A + \sin A) = c$.

14. $\tan C(a - c \cos B) = c \sin B$.

15. $c^2(\tan A - \tan B) = (a^2 - b^2)(\tan A + \tan B)$.

16. $2\Delta(\cot A + \cot B) = c^2$.

17. $a \cos A - b \cos B = \cos C(b \cos A - a \cos B)$.

18. $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

19. $(c^2 - b^2)\cos^2 A + (a^2 - c^2)\cos^2 B + (b^2 - a^2)\cos^2 C = 0$.

20. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$.

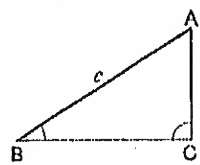
CHAPTER IX.

SOLUTION OF TRIANGLES WITH THE AID OF LOGARITHMS.

85. I. Right-angled triangles.

(i) *Given the hypotenuse* (c) *and an angle* (B).

$$A = 90^\circ - B.$$

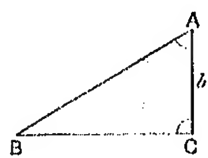


$\frac{b}{c} = \sin B$, $\therefore$ $\log b = \log c + L \sin B - 10$,

$\frac{a}{c} = \cos B$, $\therefore$ $\log a = \log c + L \cos B - 10$.

(ii) *Given a side* (b) *and an angle* (A).

$$B = 90^\circ - A.$$



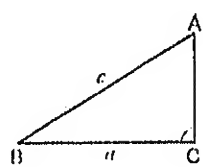
$\frac{a}{b} = \tan A$, $\therefore \log a = \log b + L \tan A - 10$,

$\frac{c}{b} = \sec A$, $\therefore \log c = \log b + L \sec A - 10$.

(iii) *Given the hypotenuse* (c) *and a side* (a).

$\sin A = \frac{a}{c}$, $\therefore L \sin A = \log a - \log c + 10$,

$B = 90^\circ - A$.



b may be found from any of the formulae

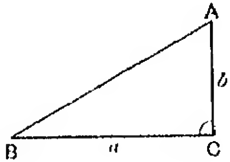
$$b = c \sin B, \qquad b = a \tan B,$$

$$b^2 = c^2 - a^2 = (c - a)(c + a).$$

The first is generally used.

(iv) *Given two sides a and b.*

$\tan B = \dfrac{b}{a}$, $\therefore$ $L \tan B = \log b - \log a + 10$,



$A = 90° - B.$

c may be found from

$$c = \frac{b}{\sin B} \quad \text{or} \quad c = \frac{a}{\cos B}.$$

The first formula is the more convenient.

[It is also possible to find c from the formula $c^2 = a^2 + b^2$, using a *Subsidiary Angle* and logarithms, thus:

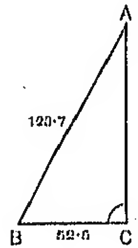
put $\tan \phi = \dfrac{b}{a}$ ....(i),

then $c^2 = a^2 + a^2 \tan^2 \phi = a^2 \sec^2 \phi$ ....(ii),

ϕ may be determined from (i) and its value substituted in (ii).]

Ex. Solve a right-angled triangle given that

$c = 123{\cdot}7$ and $a = 52{\cdot}5$.



$\sin A = \dfrac{52{\cdot}5}{123{\cdot}7}$.

$\therefore$ $L \sin A = \log 52{\cdot}5 - \log 123{\cdot}7 + 10$,

$10 + \log 52{\cdot}5 = 11{\cdot}7202$

$\log 123{\cdot}7 = 2{\cdot}0923$

$\therefore$ $L \sin A = 9{\cdot}6279$ $\quad\therefore$ $A = 25° \, 7'$,

$B = 90° - 25° \, 7' = 64° \, 53'$.

$b = c \sin B$.

$\therefore$ $\log b = \log 123{\cdot}7 + L \sin 64° \, 53' - 10$,

$\log 123{\cdot}7 = 2{\cdot}0923$,

$L \sin 64° \, 53' = 9{\cdot}9569$

$\therefore$ $\log b = 2{\cdot}0492$ $\quad\therefore$ $b = 112{\cdot}0$.

EXAMPLES XVIII.

Solve the following triangles right-angled at C, when

1. $c = 127 \cdot 2$, $B = 52° 55'$.
2. $b = 125$, $A = 37° 22'$.
3. $c = 32 \cdot 3$, $a = 16 \cdot 7$.
4. $a = 31 \cdot 3$, $b = 26 \cdot 9$.
5. $b = 122 \cdot 2$, $c = 236 \cdot 3$.
6. $c = 29 \cdot 9$, $A = 33° 22'$.
7. $b = 27 \cdot 32$, $A = 15° 17'$.
8. $c = 823 \cdot 1$, $a = 237 \cdot 5$.
9. $b = 123 \cdot 9$, $a = 321 \cdot 4$.
10. $a = 1 \cdot 732$, $B = 82° 13'$.
11. $c = 1 \cdot 532$, $B = 59° 14'$.
12. $b = 17 \cdot 32$, $a = 15 \cdot 19$.

II. Oblique-angled triangles.

86. *To solve the triangle, given the values of the three sides a, b and c.*

Method i. We may use any of the formulae

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

where $$2s = a + b + c.$$

x. The sides of a triangle are 52·8, 30·3, 73·1 feet; find ...gle opposite the smallest side. Roughly check the result ...structing a triangle with sides 5·3, 3·0, 7·2 centimetres and ...re the angles with a protractor.

...thod 1.

$$
\begin{array}{rlrl}
a = & 52{\cdot}8 & \therefore\ s = & 83{\cdot}1, \\
b = & 30{\cdot}3 & s-a = & 20{\cdot}3, \\
c = & 73{\cdot}1 & s-b = & 42{\cdot}8, \\
\hline
 & 104{\cdot}2 & s-c = & 10.
\end{array}
$$

$$\therefore\ \tan\frac{B}{2} = \sqrt{\frac{10 \times 20{\cdot}3}{83{\cdot}1 \times 42{\cdot}8}},$$

$$L\tan\frac{B}{2} = \frac{1}{2}(\log 203 - \log 83{\cdot}1 - \log 42{\cdot}8) + 10,$$

$$
\begin{array}{rl}
\log 203 = & 2{\cdot}4000 \\
 & 3{\cdot}5467 \\
\hline
2\,| & \bar{2}{\cdot}0212 \\
\hline
 & \bar{1}{\cdot}4006.
\end{array}
\qquad
\begin{array}{rl}
\log 83{\cdot}1 = & 1{\cdot}9143 \\
\log 42{\cdot}8 = & 1{\cdot}6314 \\
\hline
 & 3{\cdot}5457.
\end{array}
$$

$$\therefore\ L\tan\frac{B}{2} = 9{\cdot}4006,$$

$$\therefore\ \frac{B}{2} = 16^\circ\ 6'{\cdot}5$$

...ing to the nearest half-minute, since we afterwards have to ...to*).

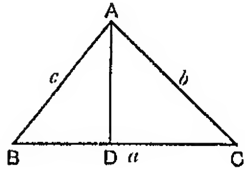
$$\therefore\ B = 32^\circ\ 13'.$$

* From the Tables tan 16° 6′ = 9·4003

diff. for 1′ = 5, ∴ diff. for ·5′ = 2·5.

...ince it is necessary to account for a diff. of 3 and this is nearer to 2·5 ...to 5, we only add on ·5′ to the angle instead of 1′.

87. *Method* ii. The following solution, depending on first principles, is due to Prof. G. H. Bryan.



Let a be the greatest of the three sides and $b > c$.

$\text{CD}^2 - \text{DB}^2 = \text{CA}^2 - \text{BA}^2 = b^2 - c^2$,

but $\text{CD} + \text{DB} = a$ ....(i).

$$\therefore \text{CD} - \text{DB} = \frac{b^2 - c^2}{a} = \frac{(b-c)(b+c)}{a},$$

$$\therefore \log(\text{CD} - \text{DB}) = \log(b-c) + \log(b+c) - \log a \quad \text{......(ii)}.$$

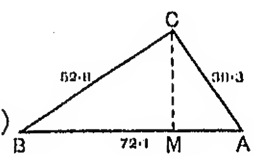
From (i) and (ii) CD and DB can be determined, thence B and C are found from

$$\cos \text{B} = \frac{\text{BD}}{c},$$

$$\cos \text{C} = \frac{\text{CD}}{b},$$

and $$\text{A} = 180^\circ - (\text{B} + \text{C}).$$

Draw a perp. on to the greatest side.



$\text{BM} + \text{MA} = 72{\cdot}1$

$\text{BM}^2 - \text{MA}^2 = \text{BC}^2 - \text{AC}^2$

$= (52{\cdot}8 + 39{\cdot}3)(52{\cdot}8 - 39{\cdot}3)$

$= 92{\cdot}1 \times 13{\cdot}5.$

$\therefore \log(\text{BM} - \text{MA}) = \log 92{\cdot}1 + \log 13{\cdot}5 - \log 72{\cdot}1,$

$\log 92{\cdot}1 = 1{\cdot}9643$

$\log 13{\cdot}5 = 1{\cdot}1303$

$3{\cdot}0946$

$\log 72{\cdot}1 = 1{\cdot}8579$

$\therefore \log(\text{BM} - \text{AM}) = 1{\cdot}2367,$

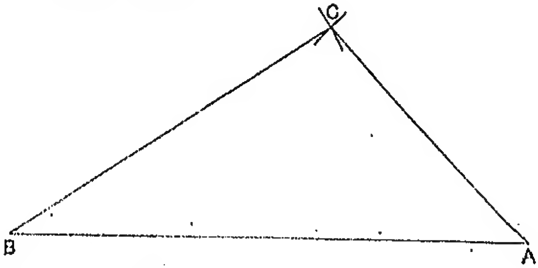
$\therefore \text{BM} - \text{AM} = 17{\cdot}25$

$\text{BM} + \text{AM} = 72{\cdot}1,$

$\therefore 2\text{BM} = 89{\cdot}35.$

But
$$\cos B = \frac{2BM}{2BC},$$
$$\therefore L\cos B = \log 89{\cdot}35 - \log 105{\cdot}6 + 10$$
$$10 + \log 89{\cdot}35 = 11{\cdot}9511$$
$$\log 105{\cdot}6 = 2{\cdot}0237$$
$$\therefore L\cos B = 9{\cdot}9274,$$
$$\therefore B = 32^\circ\ 13'.$$

88. *Rough Check.*



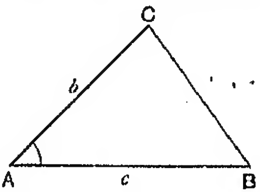
AB is drawn 7·2 centimetres long and circles described with B and A as centres and radii 5·3 and 3·9 centimetres respectively. On measuring with a protractor $\hat{CBA} = 32^\circ$.

89. *To solve a triangle, given two sides and included angle.*

Method i. Given b, c, A; $b > c$.

From Art. 84

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$



Having determined $\frac{B-C}{2}$, and knowing that $\frac{B+C}{2} = 90^\circ - \frac{A}{2}$, B and C may be found.

Also since
$$\frac{a}{\sin A} = \frac{b}{\sin B},$$
$$\therefore a = \frac{b\sin A}{\sin B}.$$

Ex. If $b = 372{\cdot}5$, $c = 395{\cdot}6$, $A = 37° 15'$, find B, C and a. Check the result by drawing a diagram as nearly as possible to scale.

Method i.

$$\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$$

$$= \frac{23{\cdot}1}{768{\cdot}1}\cot 18° 37'{\cdot}5\,^{*}.$$

$$\therefore\ L\tan\frac{C-B}{2} = \log 23{\cdot}1 + L\cot 18° 37'{\cdot}5 - \log 768{\cdot}1$$

$$\log 23{\cdot}1 = 1{\cdot}3636$$

$$L\cot 18° 37'{\cdot}5 = 10{\cdot}4723\ \dagger$$

$$11{\cdot}8359$$

$$\log 768{\cdot}1 = 2{\cdot}8855$$

$$\therefore\ L\tan\frac{C-B}{2} = 8{\cdot}9504.$$

$$\therefore\ \frac{C-B}{2} = 5° 6',$$

but
$$\frac{C+B}{2} = 90° - \frac{A}{2} = 71° 22'{\cdot}5,$$

$$\therefore\ C = 76° 28'{\cdot}5 = 76° 29'$$

$$B = 66° 16'{\cdot}5 = 66° 17' \text{ (to the nearest minute).}$$

$$a = \frac{b\sin A}{\sin B}.$$

$$\therefore\ \log a = \log 372{\cdot}5 + L\sin 37° 15' - L\sin 66° 16'{\cdot}5,$$

$$\log 372{\cdot}5 = 2{\cdot}5711$$

$$L\sin 37° 15' = 9{\cdot}7820$$

$$12{\cdot}3531$$

$$L\sin 66° 16'{\cdot}5 = 9{\cdot}9617$$

$$\therefore\ \log a = 2{\cdot}3914$$

$$\therefore\ a = 246{\cdot}2.$$

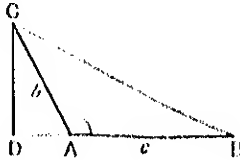
* If the number of minutes in the given angle is odd, it is advisable to retain the ·5′ in the calculations.

† $L\cot 18° 36' = 10{\cdot}4780$, the difference for $1' = 4$ and for $2' = 9$,

$\therefore$ difference for $1'{\cdot}5 = 6{\cdot}5 = 7$ (approx.).

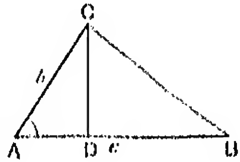
90. *Method* ii. This proof is also due to Prof. G. H. Bryan.

If A is obtuse,



$$CD = b\sin(180° - A)$$
$$AD = b\cos(180° - A)$$
$$DB = AD + c.$$

If A is acute,

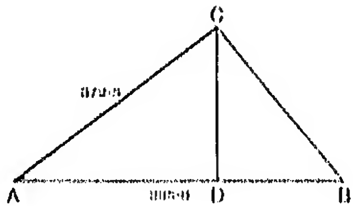


$$CD = b\sin A$$
$$AD = b\cos A$$
$$DB = c - AD.$$

Having thus determined CD and BD, we have

$$\tan B = \frac{CD}{DB},$$
$$C = 180° - (A + B)$$
$$a = \frac{CD}{\sin B}\left(\text{or } \frac{DB}{\cos B}\right).$$

Logarithms can at once be applied to all these formulae.



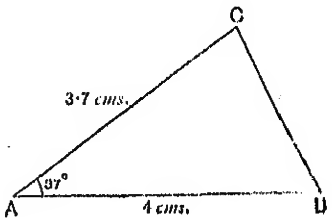
$$CD = b\sin A$$
$$\therefore \log CD = \log 372{\cdot}5 + L\sin 37° 15' - 10$$
$$\log 372{\cdot}5 = 2{\cdot}5711$$
$$L\sin 37° 15' = 9{\cdot}7820$$
$$\therefore \log CD = 2{\cdot}3531$$
$$\therefore CD = 225{\cdot}5 \text{.............................}(i).$$

$$AD = b \cos A$$
$$\therefore \log AD = \log 372{\cdot}5 + L \cos 37^\circ 15' - 10$$
$$\log 372{\cdot}5 = 2{\cdot}5711$$
$$L \cos 37^\circ 15' = 9{\cdot}9009$$
$$\therefore \log AD = 2{\cdot}4720$$
$$\therefore AD = 296{\cdot}5$$
$$\therefore DB = c - AD = 395{\cdot}6 - 296{\cdot}5$$
$$= 99{\cdot}1.$$

Now $$\tan B = \frac{CD}{DB}$$
$$\therefore L \tan B = \log 225{\cdot}5 - \log 99{\cdot}1 + 10$$
$$10 + \log 225{\cdot}5 = 12{\cdot}3531 \quad \text{from (i)}$$
$$\log 99{\cdot}1 = 1{\cdot}9961$$
$$\therefore L \tan B = 10{\cdot}3570$$
$$\therefore B = 66^\circ 16'$$
$$C = 180^\circ - (37^\circ 15' + 66^\circ 16')$$
$$= 76^\circ 29'.$$
$$a = \frac{CD}{\sin B}$$
$$\therefore \log a = \log 225{\cdot}5 - L \sin 66^\circ 16' + 10$$
$$10 + \log 225{\cdot}5 = 12{\cdot}3531$$
$$L \sin 66^\circ 16' = 9{\cdot}9616$$
$$\therefore \log a = 2{\cdot}3915$$
$$\therefore a = 246{\cdot}3.$$

91. *Rough Check.*

AB is drawn 4 cms. long and with a protractor AC is drawn so that angle CAB = 37° and AC = 3·7 cms.

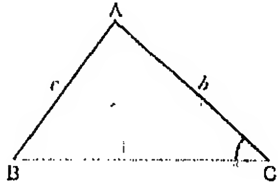


On joining CB and measuring

$$C\hat{B}A = 65^\circ$$
$$B\hat{C}A = 78^\circ$$
$$a = 2{\cdot}45 \text{ cms.} = 245.$$

92. *To solve a triangle, given two sides and an angle* (not the included angle).

Let b, c and C be the given sides and angle.



$$\frac{\sin B}{b} = \frac{\sin C}{c},$$

$$\sin B = \frac{b \sin C}{c} \quad \ldots\ldots\text{(i)}.$$

$\therefore\ L \sin B = \log b + L \sin C - \log c.$

Having determined B,

$$A = 180° - (B + C)$$

and
$$a = \frac{c \sin A}{\sin C},$$

$$\log a = \log c + L \sin A - L \sin C.$$

Equation (i) sometimes gives no value for B; sometimes gives one value, and sometimes two values, *one being the supplement of the other*.

i. If $c < b \sin C$, then $\sin B > 1$ and there is no solution.

ii. If $c = b \sin C$, then $\sin B = 1$, and $B = 90°$.

iii. If $c > b \sin C$, then $\sin B < 1$, and we have to examine whether there are one or two solutions.

(1) If $c > b$, then $C > B$, and no matter what C is, B cannot be obtuse, otherwise there would be two obtuse angles in a triangle,

$\therefore$ there is only one solution.

(2) If $c = b$, then $B = C$ and there is only one solution.

(3) If $c < b$, then $C < B$, and it is possible for B to be either acute or obtuse,

$\therefore$ there are two solutions.

This is called the **Ambiguous Case**, and it will be noticed that it only occurs when the *side opposite the given angle is less than the other given side*.

93. These results may be geometrically illustrated.

Let $\qquad ACX = \text{given angle } C,$

and $\qquad AC = \text{given side } b.$

To obtain the third angular point of the triangle a circle is described with centre A and radius $= c$; this circle may

(i) not cut CX; then there is no solution,

(ii) touch CX; then there is one solution,

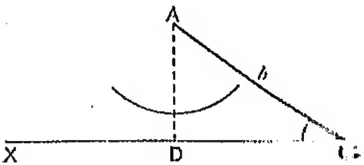
(iii) cut CX in two points; then there may be two solutions.

Draw AD perpendicular to CX,

$$AD = AC \sin ACX = b \sin C.$$

(i) If $\quad c < AD$

$$< b \sin C,$$

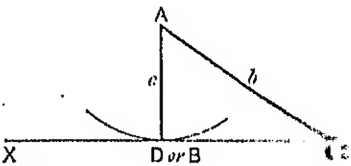


the circle does not cut CX and the third angular point cannot be found,

$\therefore$ there is no solution.

(ii) If $c = AD$

$$= b \sin C,$$



the circle touches CX at D (or B),

$$\hat{B} = 90°,$$

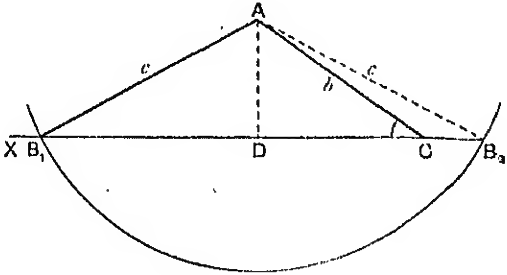
$\therefore$ there is one solution.

(iii) If $\qquad c > AD$

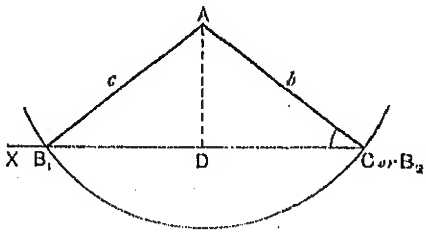
$$> b \sin C,$$

the circle cuts CX in two points B_1 and B_2 and there may be two solutions.

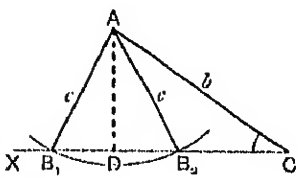
(1) If $c > b$, there is only one possible triangle AB_1C, for if AB_2C is taken, the angle ACB_2 is the supplement of given angle C, and is therefore inadmissible.



(2) If $c = b$, B_2 coincides with C and there is only one solution.



(3) If $c < b$, there are two possible triangles AB_1C and AB_2C, so that $\hat{B} = AB_1C$ or AB_2C, one value being the supplement of the other.



Ex. Given that $b = 127 \cdot 3$, $c = 59 \cdot 21$ and $C = 27° 22'$, find the other angles and side.

$$\sin B = \frac{b \sin C}{c}.$$

$$\therefore \; L \sin B = \log 127 \cdot 3 + L \sin 27° 22' - \log 59 \cdot 21,$$

$$\begin{aligned} \log 127\cdot3 &= 2\cdot1048 \\ L \sin 27° 22' &= 9\cdot6625 \\ &\overline{11\cdot7673} \\ \log 59\cdot21 &= 1\cdot7724 \\ \therefore \; L \sin B &= 9\cdot9949 \qquad \therefore \; B_1 = 81° 12' \\ & \qquad\qquad\qquad B_2 = 180° - 81° 12' = 98° 48'. \end{aligned}$$

[There is a second value of B since the side c opposite the given angle C < the other side b.]

$$A_1 = 180° - (27° 22' + 81° 12') = 71° 26',$$
$$A_2 = 180° - (27° 22' + 98° 48') = 53° 50'.$$

Also $$a = \frac{c \sin A}{\sin C}.$$

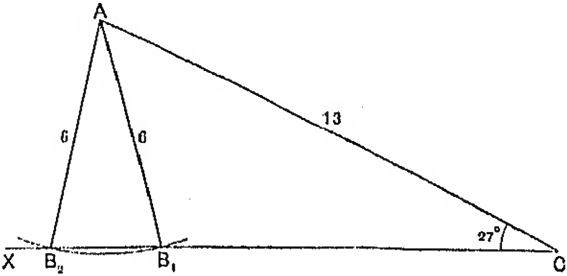
$$\therefore \; \log a_1 = \log 59\cdot21 + L \sin 71° 26' - L \sin 27° 22'$$

$$\begin{aligned} \log 59\cdot21 &= 1\cdot7724 \\ L \sin 71° 26' &= 9\cdot9768 \\ &\overline{11\cdot7492} \\ L \sin 27° 22' &= 9\cdot6625 \\ \therefore \; \log a_1 &= 2\cdot0867 \qquad \therefore \; a_1 = 122\cdot1, \end{aligned}$$

and $$\log a_2 = \log 59\cdot21 + L \sin 53° 50' - L \sin 27° 22'$$

$$\begin{aligned} \log 59\cdot21 &= 1\cdot7724 \\ L \sin 53° 50' &= 9\cdot9071 \\ &\overline{11\cdot6795} \\ L \sin 27° 22' &= 9\cdot6625 \\ \log a_2 &= 2\cdot0170 \qquad \therefore \; a_2 = 104\cdot0. \end{aligned}$$

94. *Rough Check.* By drawing a triangle with sides 13 and 6 units and angle 27°, we can not only check the above



results, but show the ambiguity. CX is drawn of indefinite length; angle XCA is made equal to 27° and CA cut off equal to 13 units. With centre A and radius 6 units a circle is drawn; since this circle is found to cut CX in two points B_1, B_2 it follows that there is an ambiguity.

Joining AB_1 and AB_2 we find that

$$A\hat{B}_2C = 78^\circ, \quad A\hat{B}_1C = 102^\circ,$$

$$CB_2 = 13 \text{ units}, \quad CB_1 = 11 \text{ units}.$$

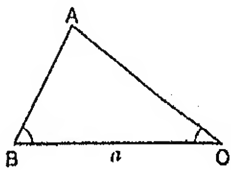
95. *To solve a triangle, given one side and two angles.*

Let a, B and C be the given values.

$$A = 180^\circ - (B + C),$$

$$b = \frac{a \sin B}{\sin A},$$

$$c = \frac{a \sin C}{\sin A}.$$



$$\therefore \log b = \log a + L \sin B - L \sin A,$$

$$\log c = \log a + L \sin C - L \sin A.$$

Ex. Given that $A = 37° 15'$, $B = 72° 5'$, $a = 75{\cdot}2$, find b and c.

$$C = 180° - (37° 15' + 72° 5') = 70° 40'.$$

$$b = \frac{a \sin B}{\sin A}.$$

$$\therefore \log b = \log 75{\cdot}2 + L \sin 72° 5' - L \sin 37° 15'$$

$$\begin{aligned} \log 75{\cdot}2 &= 1{\cdot}8762 \\ L \sin 72° 5' &= \underline{9{\cdot}9784} \\ & \quad 11{\cdot}8546 \\ L \sin 37° 15' &= \underline{9{\cdot}7820} \\ \therefore \log b &= 2{\cdot}0726 \qquad \therefore b = 118{\cdot}2. \end{aligned}$$

$$c = \frac{a \sin C}{\sin A}.$$

$$\log c = \log a + L \sin C - L \sin A$$

$$\begin{aligned} \log 75{\cdot}2 &= 1{\cdot}8762 \\ L \sin 70° 40' &= \underline{9{\cdot}9748} \\ & \quad 11{\cdot}8510 \\ L \sin 37° 15' &= \underline{9{\cdot}7820} \\ \therefore \log c &= 2{\cdot}0690 \qquad \therefore c = 117{\cdot}2. \end{aligned}$$

EXAMPLES XIX.

(*Given three sides*, Arts. 86, 87, 88.)

Find the greatest angle in the following triangles $\left(\text{using } \tan \frac{\omega}{2}\right)$, checking the results by drawing triangles roughly to scale.

1. $a = 127{\cdot}9$, $b = 32{\cdot}3$, $c = 100{\cdot}4$.
2. $a = 32{\cdot}9$, $b = 57{\cdot}31$, $c = 40{\cdot}27$.
3. $a = 52{\cdot}41$, $b = 39{\cdot}76$, $c = 25{\cdot}73$.
4. $a = 72{\cdot}41$, $b = 36{\cdot}21$, $c = 53{\cdot}2$.
5. $a = 82{\cdot}36$, $b = 72{\cdot}4$, $c = 120{\cdot}9$.
6. $a = 1075$, $b = 1021$, $c = 1572$.

Find the smallest angle

(i) by *Method* i. $\left(\text{using } \sin\frac{x}{2}\right)$, (ii) by *Method* ii.

7. $a=372\cdot4$, $b=82\cdot5$, $c=350\cdot5$.
8. $a=127\cdot9$, $b=153\cdot4$, $c=98\cdot5$.
9. $a:b:c=82\cdot3:71\cdot5:120$.
10. $a:b:c=721:432:643$.

Find all the angles $\left(\text{using } \tan\frac{x}{2}\right)$.

11. $a=15$, $b=13\cdot1$, $c=14\cdot7$.
12. $a=12\cdot72$, $b=11\cdot15$, $c=10\cdot93$.

EXAMPLES XX.

(*Given two sides and included angle*, Arts. 89, 90, 91.)

Solve the following triangles by Method i., using a formula of the type $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ and working 'half-angles' to the nearest half-minute. Check the results by drawing the figures roughly to scale.

(*Use the first of the given sides for the determination of the third side.*)

1. $b=37\cdot2$, $c=22\cdot3$, $A=29°\,38'$.
2. $a=39\cdot9$, $b=43\cdot2$, $C=38°\,14'$.
3. $b=27\cdot32$, $c=53\cdot9$, $A=58°\,38'$.
4. $a=29\cdot8$, $c=32\cdot42$, $B=26°\,14'$.
5. $a=15\cdot72$, $b=17\cdot08$, $C=37°\,25'$.
6. $b=52\cdot92$, $c=36\cdot04$, $A=62°\,17'$.

Find the remaining angles in the following triangles (i) by Method i., (ii) by Method ii. Check the results by drawing the triangles roughly to scale.

7. $a = 25{\cdot}32$, $b = 42{\cdot}9$, $C = 52° 14'$.

8. $b = 27{\cdot}51$, $c = 25{\cdot}05$, $A = 73° 12'$.

9. $a = 123{\cdot}9$, $c = 232{\cdot}4$, $B = 35° 43'$.

10. $b = 231{\cdot}2$, $c = 245{\cdot}8$, $A = 17° 22'$.

11. $a = 235{\cdot}2$, $b = 149{\cdot}7$, $C = 53° 14'$.

12. $a = 125{\cdot}9$, $c = 84{\cdot}32$, $B = 44° 28'$.

EXAMPLES XXI.

(*Given two sides and one angle, not the included angle,* Arts. 92, 93, 94.)

Point out whether the solution will be ambiguous or not in the following cases:

1. $b = 25{\cdot}9$, $c = 72{\cdot}5$, $C = 54° 15'$.

2. $a = 192{\cdot}5$, $b = 210{\cdot}2$, $A = 33° 15'$.

3. $a = 89{\cdot}2$, $c = 82{\cdot}5$, $C = 29° 13'$.

Find the remaining angles (drawing the figures to scale), when

4. $a = 82{\cdot}35$, $b = 96{\cdot}51$, $A = 55° 14'$.

5. $a = 72{\cdot}41$, $c = 65{\cdot}5$, $C = 62° 51'$.

6. $c = 421{\cdot}9$, $a = 531{\cdot}4$, $A = 72° 15'$.

7. $b = 17{\cdot}41$, $c = 19{\cdot}32$, $B = 45° 32'$.

8. $b = 15{\cdot}49$, $a = 14{\cdot}87$, $A = 35° 43'$.

9. $a = 123{\cdot}9$, $c = 172{\cdot}4$, $C = 59° 37'$.

10. $c = 12{\cdot}07$, $b = 10{\cdot}05$, $B = 37° 14'$.

Find the remaining angles and side (drawing the figures to scale), when

11. $a = 182{\cdot}5$, $b = 82{\cdot}5$, $A = 72° 15'$.

12. $b = 72{\cdot}95$, $c = 82{\cdot}31$, $B = 42° 27'$.

EXAMPLES XXII.

(*Given two angles and one side*, Art. 95.)

Find the remaining sides (checking by diagrams), when

1. $A=15^\circ\,42'$, $B=55^\circ\,17'$, $c=123\cdot4$.
2. $A=35^\circ\,17'$, $C=45^\circ\,13'$, $b=\ 42\cdot1$.
3. $B=45^\circ\,15'$, $C=72^\circ\,12'$, $a=\ 39\cdot05$.
4. $A=72^\circ\,13'$, $C=54^\circ\,22'$, $a=\ 17\cdot21$.
5. $A=85^\circ\,25'$, $B=42^\circ\,13'$, $b=\ 18\cdot95$.
6. $C=84^\circ\,37'$, $B=43^\circ\,17'$, $c=\ 54\cdot27$.
7. $A=54^\circ\,43'$, $C=42^\circ\,39'$, $b=\ 72\cdot45$.
8. $B=64^\circ\,23'$, $C=72^\circ\,43'$, $a=\ 18\cdot92$.
9. $A=54^\circ\,33'$, $B=49^\circ\,22'$, $a=124\cdot5$.
10. $A=62^\circ\,21'$, $C=54^\circ\,37'$, $c=721\cdot6$.

EXAMPLES XXIII.

(*Miscellaneous.*)

1. If $a=15\cdot7$, $b=16\cdot4$, $c=19\cdot7$, find the smallest angle $\left(\text{using } \tan\dfrac{X}{2}\right)$.

2. Given that two sides of a triangle are 17·8 and 18·9 and the included angle 53° 28′, find the remaining angles.

3. In which of the following triangles are there ambiguous solutions?

$$b=17\cdot5,\quad c=15\cdot2,\quad C=45^\circ\,22'.$$
$$a=14\cdot25,\quad b=17\cdot5,\quad A=33^\circ\,17'.$$
$$c=18\cdot9,\quad a=10\cdot4,\quad C=65^\circ\,17'.$$

4. Solve the triangle, given $b=18\cdot42$, $c=14\cdot39$, $C=48^\circ\,54'$.

5. If $A=75^\circ\,14'$, $B=32^\circ\,13'$ and $a=17\cdot42$, find the remaining sides.

6. If two sides of a triangle are 52·44 and 36·92 and included angle 72° 38′, find the remaining angles.

7. Given that the three sides of a triangle are 124·2, 82 43·2, find the greatest angle $\left(\text{using } \tan\frac{X}{2}\right)$.

8. If $a=18{\cdot}2$, $b=20{\cdot}4$ and A = 44° 17′, find B and C.

9. Given that B = 39° 14′, C = 42° 15′ and $a=123{\cdot}9$, find remaining sides.

10. If $a=19{\cdot}45$, $b=21{\cdot}32$ and A = 35° 14′, solve the trian

11. If $b=87{\cdot}9$, $c=94{\cdot}7$ and A = 17° 15′, find B by Method Art. 90.

12. Given that the three sides of a triangle are 42·1, and 82·9, find the smallest angle by Method ii. Art. 87.

13. Given that $a=28{\cdot}92$, $b=14{\cdot}75$ and A = 63° 15′, find remaining angles.

14. If A = 49° 15′, B = 71° 16′ and $c=42{\cdot}17$, find a and b.

15. Given that the two sides and included angle are 2 19·7 and 48° 32′, solve the triangle.

16. The three sides of a triangle are 27·42, 52·45 and 3 find the greatest angle $\left(\text{using } \tan\frac{X}{2}\right)$.

17. If A = 37° 15′, C = 49° 39′ and $a=197{\cdot}4$, find b and c.

18. Given that $b=19{\cdot}45$, $c=17{\cdot}32$ and B = 72° 15′, A and C.

19. If the three sides of a triangle are 542·3, 672·4 823·6, find the angles by Method ii. Art. 87.

20. The two sides and included angle of a triangle are 8 68·7 and 53° 26′, find the remaining angles.

21. If B = 108° 5′, C = 14° 53′ and $a=18{\cdot}95$, find b and c.

22. If $a=15{\cdot}2$, $c=18{\cdot}9$ and B = 107° 15′, find A by Method Art. 90.

23. Given that $b=24{\cdot}45$, $c=26{\cdot}92$ and B = 40° 28′, find remaining angles.

24. $A=62^\circ 15'$, $B=49^\circ 37'$ and $c=18\cdot41$; find a and b.

25. If $b=83\cdot5$, $c=182\cdot4$ and $A=48^\circ 22'$, find a.

26. Half the difference between the base angles of a triangle is $10^\circ 14'$ and the sides adjacent to the base are proportional to $18\cdot42$ and $16\cdot35$ respectively; find the vertical angle.

27. If the hypotenuse of a right-angled triangle is $129\cdot6$ and one of the angles $35^\circ 19'$, find the remaining sides.

28. If one of the base angles of a triangle is $49^\circ 15'$ and the adjacent side $128\cdot4$, find the altitude of the triangle which is drawn from the extremity of the given side.

29. Given that the three sides of a triangle are $60\cdot4$, $100\cdot8$ and $129\cdot6$, find the length of the perpendicular from the greatest angle to the opposite side.

30. If the two sides and included (vertical) angle of a triangle are $107\cdot5$, $130\cdot4$ and $62^\circ 14'$, find the altitude.

Miscellaneous Examples on Chapters VIII and IX start in Test Paper XXIX, page 223.

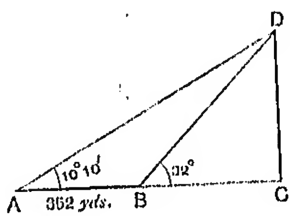
CHAPTER X.

HEIGHTS AND DISTANCES.

To find the height and distance of an inaccessible object using two points of observation.

96. 1. *When the two points* A *and* B *are at a known distance apart on a horizontal plane and in the same vertical plane as the object.*

Ex. Walking towards a tower at 4 miles an hour, a man observed at one time that the angle of elevation of its top was 10° 10′, and three minutes afterwards that it was 32°. Find its height and distance from the second point of observation.



AB = distance walked in 3 mins.

$= \frac{1}{5}$ mile = 352 yds.

We have now to **link** DC and AB together by means of some other line; *it is convenient to use one of the lines joining their extremities.*

$$\frac{DC}{352} = \frac{DC}{DB} \cdot \frac{DB}{352}$$

$$= \frac{\sin DBC}{1} \cdot \frac{\sin DAB}{\sin ADB},$$

$$\therefore \ DC = \frac{352 \sin 32^\circ \cdot \sin 10^\circ 10'}{\sin 21^\circ 50'}.$$

$$\log DC = \log 352 + L\sin 32° + L\sin 10°\ 10' - L\sin 21°\ 50' - 10$$

$\log 352 = 2{\cdot}5465$

$L\sin 32° = 9{\cdot}7242$

$L\sin 10°\ 10' = 9{\cdot}2466$

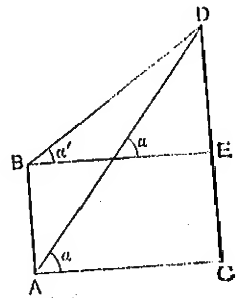
$\qquad\qquad 21{\cdot}5173$

$L\sin 21°\ 50' = 9{\cdot}5704$

$\therefore\ \log DC = 1{\cdot}9469 \qquad\qquad \therefore\ DC = 88{\cdot}49$ yds.

BC may be found from $BC = DC \cot 32°$.

97. 2. *When the two points A and B are at a known distance apart in a vertical line and in the same vertical plane as the object.*



Here we *link* DC and AB by means of DA,

$$\frac{DC}{BA} = \frac{DC}{DA} \cdot \frac{DA}{BA} = \frac{\sin DAC}{1} \cdot \frac{\sin DBA}{\sin BDA},$$

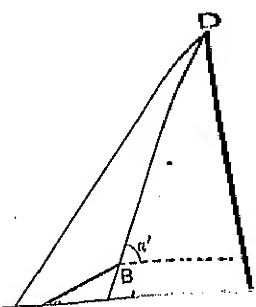
where $\qquad BDA = \alpha - \alpha'$

$\qquad\qquad DBA = 90° + \alpha'$.

$$\therefore\ DC = \frac{BA \sin\alpha \cos\alpha'}{\sin(\alpha - \alpha')},$$

and then $AC = DC \cot\alpha$.

98. 3. *When the two points A and B are at a known distance apart along a line inclined to the horizontal, but in the same vertical plane as the object.*



Linking together DC and AB by means of AD,

$$\frac{DC}{AB} = \frac{DC}{AD} \cdot \frac{AD}{AB} = \frac{\sin DAC}{1} \cdot \frac{\sin ABD}{\sin ADB},$$

where $\qquad ABD = 180° - ABE$

$\qquad\qquad\quad = 180° - (\alpha' - \beta)$

$\qquad\qquad ADB = \alpha' - \alpha$.

$$\therefore\ DC = \frac{AB \sin\alpha \sin(\alpha' - \beta)}{\sin(\alpha' - \alpha)},$$

EXAMPLES XXIV.

1. A man walking towards a tower observes that the angle of elevation of the top is 11° 20′ and on going 55 metres nearer finds it to be 14° 35′. Find the height of the tower in metres.

2. The angle subtended by 2 blockhouses at a certain point is 35° and on walking 5 miles towards one the angle is found to be 58° 30′; what is then the distance of the person from the second?

3. From the bottom of a tower 250 feet high the angle of elevation of the summit of a mountain is found to be 15° 20′ and from the top 14° 15′. Find the height of the mountain in feet.

4. The angles of depression of 2 boats in the same vertical plane as the observer (from the top of a cliff 180 feet high) are 48° 30′ and 32° 25′; find the distance apart of the boats.

5. At a point on the same level as the foot of a tower, the tower subtends an angle of 25° 45′, while a flagstaff 55 feet high on the top of the tower subtends an angle of 6° 20′. Find the height of the tower in feet.

6. An observer in a balloon finds that the angle of depression of a fort is 28° 15′ and on descending vertically through 580 feet, finds the angle to be 12° 10′. Find the height of the balloon at the first observation and the horizontal distance of the fort from the point of ascent, assuming that the balloon has only been moving vertically.

7. The angle of elevation of the summit of a hill is 10° 15′ and on walking 1000 yds. up an incline of 7° 30′ it is found to be 15° 40′. Find the height of the hill.

8. From a point at the foot of the mountain, the elevation of the observatory on Ben Nevis is 58° 15′, and a man after walking 2000 feet up a slope of 32° finds that the elevation is 73°. Find the approximate height of Ben Nevis.

9. A pole leans over 15° towards the S. and from a point 50 metres to the N. the angle of elevation of the summit is 10° while from a certain point to the S. the angle of elevation of the [illegible]

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10. A man wishes to find the breadth of a river, so from one bank he measures the elevation of the top of a tower on the opposite bank and finds it to be 53° 25′. He then recedes 35 feet and finds that the elevation is 46° 35′. Calculate the breadth of the river from these observations.

11. A and B are two stations 1200 yds. apart on the shore running from East to West. At A a lighthouse bears S. 43° 20′ W. and at B it bears S. 33° E. Find the distance of the lighthouse from the shore.

12. The angle of elevation of a cliff from one point of observation is 11° 35′ and from a second point of observation 820 yds. nearer to the cliff it is 65° 15′. Find the height of the cliff.

13. A balloon is vertically over a point which lies in a direct line between two observers who are 2000 feet apart, and who note the angles of elevation of the balloon to be 34° 15′ and 59° 25′; find its height.

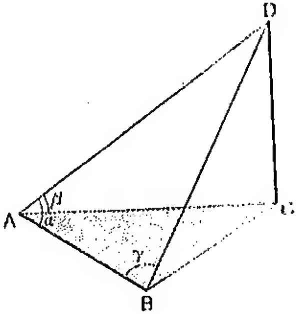
14. A and B are two points on one bank of a straight river, distant from one another 645 yards; C is on the other bank and the angles CAB, CBA are respectively 48° 31′ and 75° 25′; find the width of the river.

15. From the top A of a cliff 590 feet high, the angle of elevation of a balloon B was observed to be 46° 35′ and the angle of depression of its shadow S upon the sea 62°. A, B and S being in the same vertical plane, and the altitude of the sun being 64° 31′, find the height of the balloon above sea-level.

16. A tower standing on the edge of a cliff is viewed by a man lying down on the shore. The tower and the cliff immediately under the tower are found to subtend each an angle of 29° 15′, while the distance of the eye from a point at the foot of that part of the cliff is 22 feet. Find the height of the tower.

17. The altitude of a certain rock is observed to be 45° 35′ and after walking 875 feet towards the rock up a slope inclined at an angle of 31° 25′ to the horizon, the observer finds that the altitude is 75° 45′. Find the vertical height of the rock above the first point of observation.

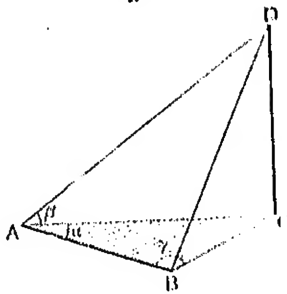
To find the height of an inaccessible object when the two points of observation A *and* B *in a horizontal plane are not in the same vertical plane as the object.*



99. 1. The angle of elevation DAC at the first point of observation A, also DAB and DBA are measured,

$$\frac{DC}{AB} = \frac{DC}{AD} \cdot \frac{AD}{AB} = \frac{\sin\beta}{1} \cdot \frac{\sin\gamma}{\sin(\alpha+\gamma)},$$

$$\therefore DC = \frac{AB \sin\beta \sin\gamma}{\sin(\alpha+\gamma)}.$$



100. 2. The angle of elevation DAC at the first point of observation A and the two angles CAB and ABC are measured,

$$\frac{DC}{AB} = \frac{DC}{AC} \cdot \frac{AC}{AB} = \frac{\tan\beta}{1} \cdot \frac{\sin\gamma}{\sin(\alpha+\gamma)},$$

$$\therefore DC = \frac{AB \tan\beta \sin\gamma}{\sin(\alpha+\gamma)}.$$

101. 3. The angles of elevation of the object at the two points of observation are measured, AB being perpendicular to AC.

Ex. From a point A the angle of elevation of the summit of a mountain due N. is 15° 20′ and walking 5 miles due E. the angle of elevation of the summit is found to be 11° 25′. Find the height of the mountain.

$$AC = DC \cot 15^\circ\, 20'$$

$$BC = DC \cot 11^\circ\, 25'.$$

But $$BC^2 - AC^2 = AB^2 = 25,$$

$$\therefore DC^2 (\cot^2 11^\circ\, 25' - \cot^2 15^\circ\, 20') = 25,$$

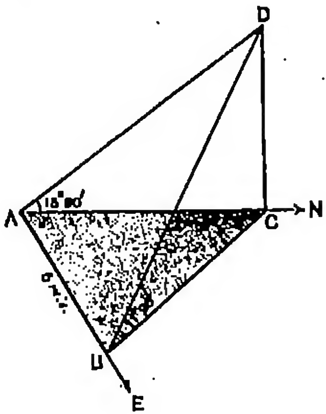
... the angle of elevation of the

or

$$DO = \frac{5}{\sqrt{(\cot 11^\circ 25' + \cot 15^\circ 20')(\cot 11^\circ 25' - \cot 15^\circ 20')}}$$

$$= \frac{5}{\sqrt{(4{\cdot}9520 + 3{\cdot}6472)(4{\cdot}9520 - 3{\cdot}6472)}}$$

$$= \frac{5}{\sqrt{8{\cdot}5992 \times 1{\cdot}3048}};$$



$\therefore \log DO = \log 5 - \frac{1}{2}[\log 8{\cdot}5992 + \log 1{\cdot}3048]$

$\log 5 = {\cdot}6990$ $\qquad$ $\log 8{\cdot}5992 = {\cdot}9345$

${\cdot}5250$ $\qquad$ $\log 1{\cdot}3048 = {\cdot}1155$

$\therefore \log DO = {\cdot}1740,$ $\qquad$ $2\,|\,1{\cdot}0500$

${\cdot}5250$

$\therefore DO = 1{\cdot}493$ miles.

EXAMPLES XXV.

1. From one point of observation, the elevation of the top of an object is 23° 10′ and the lines joining the top of the object to the two points of observation, make with the line joining the two points of observation angles of 62° 14′ and 67° 25′. If the distance between the two points is 850 feet, find the height of the object.

2. An object is seen due North at an elevation of 35° 33' and on walking 1050 yards N. 15° E. it is found that the vertical plane through the observer and object makes an angle of 108° 30' with the direction the observer has walked. Find the height of the object.

3. The angle of elevation of the summit of a mountain due N. is 14° 27' and on walking 7000 yards due W. it is found to be 10° 24'. Find the height of the mountain.

4. The angle of elevation of a balloon from a station due S. of it is 45° 35' and from another station 725 feet due W. of the former, the elevation is 40° 22'. Find the height of the balloon.

5. A column is E. 17° 30' S. of an observer and at noon the end of the shadow is N.E. of him. If the shadow is 75 feet long and the elevation of the top of the column is 45°, find the height of the column.

6. From the extremities A and B of a horizontal base-line 1200 ft. in length, it is found that the angles CAB and CBA are 67° 30' and 49° 15', C being the foot of a tower. From A the elevation of the top of the tower is 8° 17'; find the height of the tower.

7. A and B are two points 1500 feet apart and D is the top of a tower. The angles DAB and DBA are 59° 15' and 54° 30' respectively. The elevation of a tower from A being 5° 15', find its height.

8. A person at A observes the elevation of a flagstaff DC to be 68° 10' and on walking at right angles to AC (where C is the bottom of the flagstaff) a distance 95 feet to a point B, finds that the elevation is now 47° 15'. Find the height of the flagstaff.

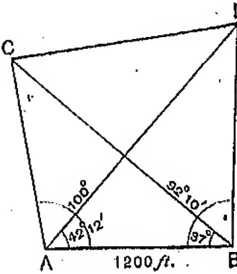
9. From one point on a river's bank the elevation of a tower 450 feet high on the same bank is 55°, and from a point on the other bank exactly opposite the first, the elevation is 42° 30'. Find the breadth of the river.

10. A and B are two points in the same horizontal plane 1250 feet apart and the angle of elevation of a tower DC as seen from A is 11° 24', and the angles DAB and DBA are 64° 21' and 47° 15' respectively. Find the height of the tower, D being its top.

MISCELLANEOUS EXAMPLES.

102. To find the distance between two inaccessible objects on a horizontal plane.

Ex. 1. Two forts C and D are observed from places A and B, 1200 ft. apart, and it is found that $C\hat{A}B = 100°$, $D\hat{A}B = 42° 12'$, $C\hat{B}A = 37°$, $D\hat{B}A = 92° 10'$. Find the distance between the forts.



$$A\hat{D}B = 180° - 92° 10' - 42° 12'$$
$$= 45° 38'$$
$$A\hat{C}B = 180° - 100° - 37°$$
$$= 43°.$$

From $\triangle$ DAB,

$$AD = \frac{1200 \sin 92° 10'}{\sin 45° 38'};$$

from $\triangle$ CAB,

$$CA = \frac{1200 \sin 37°}{\sin 43°},$$

$$\therefore \log AD = \log 1200 + L \sin 92° 10' - L \sin 45° 38'$$

$$\log 1200 = 3{\cdot}0792$$
$$L \sin 92° 10' = 9{\cdot}9997$$
$$13{\cdot}0789$$
$$L \sin 45° 38' = 9{\cdot}8542$$
$$\therefore \log AD = 3{\cdot}2247 \qquad \therefore AD = 1678 \text{ ft.}$$

Also $\log CA = \log 1200 + L \sin 37° - L \sin 43°$

$$\log 1200 = 3{\cdot}0792$$
$$L \sin 37° = 9{\cdot}7795$$
$$12{\cdot}8587$$
$$L \sin 43° = 9{\cdot}8338$$
$$\therefore \log CA = 3{\cdot}0249 \qquad \therefore CA = 1059 \text{ ft.}$$

In the $\triangle$ CAD we now know two sides and the included angle and thus can calculate the third side CD as in Art. 89.

$$\tan\frac{ACD-ADC}{2}=\frac{619}{2737}\cot 28^\circ 54',$$

$$\log 619 = 2{\cdot}7917$$
$$L\cot 28^\circ 54' = 10{\cdot}2580$$
$$13{\cdot}0497$$
$$\log 2737 = 3{\cdot}4373$$

$$\therefore\ L\tan\frac{ACD-ADC}{2}=9{\cdot}6124,$$

$$\therefore\ \frac{ACD-ADC}{2}=22^\circ 16'{\cdot}5.$$

$$\text{But } \frac{ACD+ADC}{2}=61^\circ 6',$$

$$\therefore\ A\hat{C}D=83^\circ 22'{\cdot}5.$$

$$\text{Now CD}=\frac{AD\sin CAD}{\sin ACD}$$
$$=\frac{1678\sin 57^\circ 48'}{\sin 83^\circ 22'{\cdot}5},$$

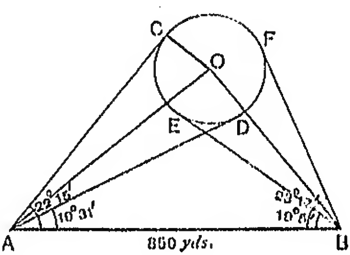
$$\log 1678 = 3{\cdot}2247$$
$$L\sin 57^\circ 48' = 9{\cdot}9275$$
$$13{\cdot}1522$$
$$L\sin 83^\circ 22'{\cdot}5 = 9{\cdot}9970$$
$$\therefore\ \log CD = 3{\cdot}1552$$
$$\therefore\ CD = 1430 \text{ feet}.$$

Ex. 2. A circular tower is observed from points A and B 850 yards apart, and on measuring the angles CAB, DAB, FBA and EBA they are found to be

22° 15′, 19° 31′, 23° 17′ and 19° 5′

respectively.

Find the radius of the tower.



$$\frac{OC}{850}=\frac{OC}{OA}\cdot\frac{OA}{850}$$

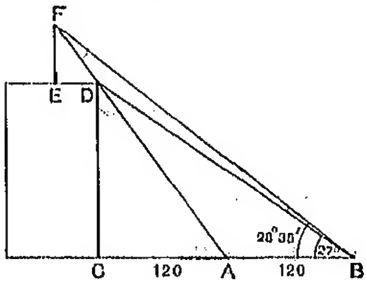
$$=\frac{\sin 1^\circ 22'\,.\,\sin 21^\circ 11'}{\sin 137^\circ 56'}.$$

$$[O\hat{A}C=\tfrac{1}{2}(CAB-DAB)$$
$$O\hat{B}A=\tfrac{1}{2}(EBA+FBA)$$
$$O\hat{A}B=\tfrac{1}{2}(DAB+CAB)]$$

$$\therefore \log OC = \log 850 + L\sin 1°\,22' + L\sin 21°\,11' - L\sin 42°\,4' - 10,$$

$$\begin{aligned} \log 850 &= 2{\cdot}9294 \\ L\sin 1°\,22' &= 8{\cdot}3880 \\ L\sin 21°\,11' &= \underline{9{\cdot}5579} \\ & 20{\cdot}8753 \\ L\sin 42°\,4' &= \underline{9{\cdot}8261} \\ \therefore \log OC &= 1{\cdot}0492 \\ \therefore OC &= 11{\cdot}2 \text{ yards.} \end{aligned}$$

Ex. 3. A flagstaff is placed at the middle of the top of a square tower and a person standing on the ground opposite the middle of a side of the tower and distant 120 feet from it just sees the flagstaff; on receding another 120 feet the elevations of the top of the tower and flagstaff are 27° and 28° 35′. Find the heights of the tower and flagstaff.



$$\frac{DC}{240} = \tan 27°,$$

$$\therefore DC = 240 \times {\cdot}5095$$
$$= 122{\cdot}28 \text{ feet.}$$

$$\left[\cot ODA = \frac{DC}{120} = 2\tan 27° = 1{\cdot}0190,\right.$$

$$\therefore O\hat{D}A = 44°\,28'$$

$$F\hat{B}D = 1°\,35'$$

$$\begin{aligned} D\hat{F}B &= BDA - FBD \\ &= BDC - ODA - FBD \\ &= 63° - 44°\,28' - 1°\,35' \\ &= 16°\,57' \end{aligned}$$

$$\left. D\hat{A}B = 90° + 44°\,28' = 134°\,28'.\right]$$

Now $$\frac{FE}{DC}=\frac{FD}{DA}=\frac{FD}{DB}\cdot\frac{DB}{DA}=\frac{\sin FBD\,.\,\sin DAB}{\sin DFB\,.\,\sin DBA}$$

$$=\frac{\sin 1^\circ 35'\,.\,\sin 134^\circ 28'}{\sin 16^\circ 57'\,.\,\sin 27^\circ};$$

$$\therefore\ \log FE=\log 122{\cdot}28+L\sin 1^\circ 35'+L\sin 45^\circ 32' - L\sin 16^\circ 57' - L\sin 27^\circ$$

$\log 122{\cdot}28=2{\cdot}0874$

$L\sin 1^\circ 35'=8{\cdot}4459$

$L\sin 45^\circ 32'=9{\cdot}8534$

$20{\cdot}3867$

$19{\cdot}1217$

$\therefore\ \log FE=1{\cdot}2650$

$L\sin 16^\circ 57'=9{\cdot}4647$

$L\sin 27^\circ=9{\cdot}6570$

$19{\cdot}1217$

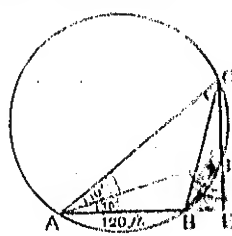
$\therefore\ FE=18{\cdot}41$ feet.

Ex. 4. A flagstaff is fixed on the top of a tower and an observer finds that the angles subtended by the flagstaff and tower at a point on the horizontal plane through the bottom of the tower are 10° and 15°. On walking a distance of 120 feet towards the tower, he finds that the flagstaff again subtends an angle of 10°. Find the heights of the flagstaff and tower.

CD and DE represent the flagstaff and tower respectively.

Since $C\hat{A}D=C\hat{B}D$,

the four points CABD are concyclic.



$$A\hat{D}B=ADE-BDE$$
$$=90^\circ-DAE-CAB$$
$$=90^\circ-15^\circ-25^\circ$$
$$=50^\circ.$$
$$A\hat{B}D=180^\circ-(BAD+ADB)$$
$$=115^\circ.$$

$$\frac{CD}{AB}=\frac{\sin CAD}{\sin ADB}\quad\text{(Art. 73)}$$

$$=\frac{\sin 10^\circ}{\sin 50^\circ};$$

$$\therefore \log \text{CD} = \log 120 + L \sin 10^\circ - L \sin 50^\circ$$

$$\begin{aligned} \log 120 &= 2{\cdot}0792 \\ L \sin 10^\circ &= 9{\cdot}2397 \\ \hline & \quad 11{\cdot}3189 \\ L \sin 50^\circ &= 9{\cdot}8843 \\ \hline \therefore \log \text{CD} &= 1{\cdot}4346 \end{aligned} \qquad \therefore \text{CD} = 27{\cdot}20 \text{ feet.}$$

$$\frac{\text{DE}}{\text{AB}} = \frac{\text{DE}}{\text{DB}} \cdot \frac{\text{DB}}{\text{AB}} = \frac{\sin \text{DBE} \cdot \sin \text{DAB}}{\sin \text{ADB}}$$

$$= \frac{\sin 65^\circ \cdot \sin 15^\circ}{\sin 50^\circ}$$

$$\therefore \log \text{DE} = \log 120 + L \sin 65^\circ + L \sin 15^\circ - L \sin 50^\circ - 10$$

$$\begin{aligned} \log 120 &= 2{\cdot}0792 \\ L \sin 65^\circ &= 9{\cdot}9573 \\ L \sin 15^\circ &= 9{\cdot}4130 \\ \hline & \quad 21{\cdot}4495 \\ L \sin 50^\circ &= 9{\cdot}8843 \\ \hline \therefore \log \text{DE} &= 1{\cdot}5652 \end{aligned} \qquad \therefore \text{DE} = 36{\cdot}75 \text{ feet.}$$

EXAMPLES XXVI.

(Miscellaneous.)

1. The elevation of a tower from a point A due S. of it is observed to be 43° 18′, and from a point B due E. of A to be 28° 30′. If AB = 240 ft. find the height of the tower.

2. A man stands on the top of a wall of height h feet and observes the elevation of a telegraph post to be α, he then descends from the wall and finds the elevation to be β, show that the height of the post exceeds that of the man by

$$h + \frac{h \sin \alpha \cos \beta}{\sin (\beta - \alpha)} \text{ feet.}$$

3. In order to determine the height of a mountain a base was measured of 2750 feet. At either extremity of the base were taken the angles formed by the summit and the other extremity. These angles were 59° 15′ and 112° 52′. Also at the extremity from which the latter angle was taken the angular height of the mountain was 12° 14′. Find the height of the mountain.

4. Two forts P and Q are observed from two stations A and B, 1350 yards apart, and it is found that $P\hat{A}B = 108°$, $Q\hat{A}B = 43° 12'$, $P\hat{B}A = 32° 10'$ and $Q\hat{B}A = 87° 12'$. Calculate the distance between the forts.

5. A man standing on one bank of a straight river sees two objects on the further side, and the lines joining his position to them make with the direction of flow of the river angles of 50° 13′ and 70° 15′. He walks down stream until the objects are seen in line, and finds that the line joining his position to them now makes an angle of 103° 52′ with the direction of flow of the river. He measures the distance he has walked, and finds it is 150 yards. What is the distance between the objects?

6. Two persons, P, Q, 1000 yards apart, stationed on a coast which runs East and West, observe a ship when it is due N. of P, and again when it is due N. of Q. In the former case it is 43° 10′ West of North as seen from Q, in the latter case it is 28° 30′ East of North as seen from P. Determine the direction in which the ship is travelling.

7. An observer wishing to determine the length of an object in the horizontal plane through his eye, finds that the object subtends the angle α at his eye when he is in a certain position A. He then finds two other positions B and C where the object subtends the same angle α. Express the length of the object in terms of the sides and area of the triangle ABC and of the angle α.

8. Two stations A and B are in the same vertical plane with C and on opposite sides of it. The altitude of A above the horizontal plane through C is 175 ft. and the inclinations of AC, BC, AB to the vertical are observed to be 43°, 21° 15′ and 35° 36′. Find the altitude of B above C.

9. A ship sailing due N. observes two lighthouses bearing N. 43° E. and N. 22° 10′ E. After sailing 18 miles the lighthouses are seen to be in a line due E., find the distance between the lighthouses.

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10. A person on a ship sailing north sees two lighthouses, which are 14 miles apart, in a line due west; after one hour's sailing one of them bears 22° to the west of south and the other 47° 15′ to the west of south. Find the ship's rate.

11. A flagstaff at the top of a tower subtends an angle 8° 15′ at two stations in a horizontal plane passing through the foot of the tower distant 63 feet apart, the tower and stations being in the same vertical plane. The angle subtended by the tower at the farther point of observation being 17° 10′, find the height of the flagstaff.

12. Two objects situated in a line running north and south are separated by a river. A person walks from the southern object 135 feet in a direction W. 24° 10′ S. and then finds that the line joining the objects subtends an angle of 17° 15′ at his eye. Find the distance between the objects.

13. A tower on the side of a hill subtends equal angles at two points of observation in the same horizontal plane. If the elevations of the top of the tower from the two points are 66° and 60° 15′ and the distance between the points 95 feet, find the height of the tower.

14. A ship 320 feet long is at anchor with her bow due south of a lighthouse. When she is lying due E. and W., the elevation of the lighthouse is observed from the stern of the ship to be 14° 30′ and the horizontal angle between the line from the stern to the lighthouse and the direction of the ship 58° 12′. Find the height of the lighthouse above the level of the deck and the distance of the ship's bow from the lighthouse.

15. A and B are consecutive milestones on a straight road running E. and W., and a distant spire is seen from A in a direction N. 22° W. and from B in a direction N. 35° E. Find the shortest distance of the spire from the road.

16. A flagstaff on the top of a tower subtends its greatest angle 11° 14′ at a point in the horizontal plane through the foot of the tower and at a distance 220 feet from the tower. Find the heights of the tower and flagstaff.

17. The angular elevation of a tower at a place A due south of it is 28° 30′ and at a place B due west of A, and at a distance 225 feet from it, the elevation is 17° 12′. Find the height of the tower.

18. Two poles a and b feet long respectively are placed vertically in a horizontal plane so that each subtends an angle of 30° at a point in the line joining their feet. If β and β' be the angles which they subtend at any point in the horizontal plane at which the line joining their feet subtends a right angle, show that

$$3(a+b)^2 = a^2 \cot^2\beta + b^2 \cot^2\beta'.$$

19. A flagstaff, 35 feet high, standing on the edge of a cliff, subtends an angle of 3° 10′ at a ship at sea, the angle of elevation of the cliff being 26° 30′. Find the distance of the base of the cliff from the ship.

20. A statue on the top of a pillar, standing on level ground, is found to subtend the greatest angle 28° 14′ at the eye of an observer when his distance from the pillar is 75 feet. Find the height of the pillar.

21. A tower stands on an inclined plane whose inclination to the horizon is 8° 17′, and from a point 120 feet down the plane, the tower subtends an angle of 57° 12′. Find the height of the tower.

22. A person walking along a road finds that two objects appear to be in the same straight line making an angle of 42° with the direction in which he is walking. On proceeding 290 feet he finds that the line joining the objects subtends its greatest angle, and that the line to the nearer object makes an angle of 37° with the direction from which he has come. Find the distance between the objects.

23. The line of greatest slope of the plane side of a hill runs downwards from W. to E., and is inclined to the horizon at an angle of 32°. It is required to construct a straight railway on it inclined at an angle 2° 30′ to the horizon. Show that the point of the compass towards which it will be directed is N. 4° W.

24. A flagstaff 6 ft. high stands on the top of a pyramid with a square base and the extremity of the shadow just reaches one of the sides and is distant 32 ft. and 24 ft. from the ends of that side. If the elevation of the sun is 54°, find the height of the pyramid.

25. Two spectators, at two stations 100 ft. apart, observe, at the same instant, the altitude of a kite, and find it to be 38° 15′ at each place. The angle which the line joining one station and the kite subtends at the second station is 57° 10′. Find the height of the kite at the moment of observation.

26. At each end of a line 1000 yards long, the elevation of the summit of a tower is 8° 24′, and at the middle point of the line the elevation is 10° 30′. Find the height of the tower.

27. A spherical ball of diameter 15 feet subtends an angle of 22° at a man's eye and the elevation of its centre is 38° 10′. Find the height of the centre of the ball.

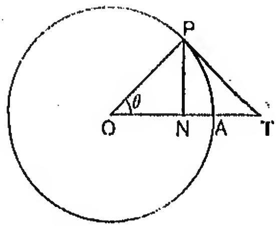
28. A and B are the summits of two mountains which rise from a horizontal plain, B being 1000 ft. above the plain. The angle of elevation of A as seen from a point C in the plain (in the same vertical plane with A and B) is 50°, while the angle of depression of C, viewed from B, is 28° 58′, and the angle subtended by AC at B is 50°. Find the height of A (C being between A and B).

29. The mutual distances of three points in a horizontal plane, from which the elevations of an inaccessible object are the same, are 732, 820 and 924 yards. Find the height of the object, its elevation from each of the three stations being 36°.

The Dip of the Horizon.

103. *Lemma.* If θ is the circular measure of an angle, then when θ is indefinitely small $\sin\theta = \theta = \tan\theta$, approx.

In a circle of unit radius, let POA be an angle whose circular measure is θ, PT the tangent at P and PN perpendicular to OA. Then



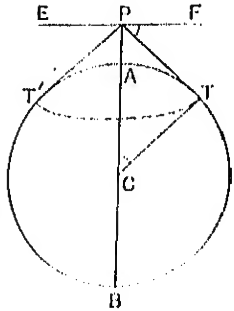
$$\sin\theta = PN$$
$$\theta = \text{arc } PA$$
$$\tan\theta = PT.$$

Now as P approaches A, N and T approach A from opposite sides and consequently PN, the arc PA and PT get nearer and nearer in value, until when θ is indefinitely small they only differ from one another by an indefinitely small quantity.

Thus approximately $\sin\theta = \theta = \tan\theta$,

when θ is indefinitely small.

[A more rigorous proof of this result is given in Chapter XX.]



104. If P is a point above the earth's surface and tangents be drawn from P to the earth they will obviously touch the earth in a circle (part of which is represented by the dotted line in the figure).

This circle is called the *Visible Horizon.* If a horizontal plane EPF be drawn through the observer at P then any one of the angles TPF is called the *Dip of the Horizon* and PT (= AT approx.) the *Distance of the Horizon.* If r is the radius of the earth, and PA $= h$,

$$PT^2 = PA \, . \, PB$$

$$= h(h + 2r).$$

Since h is small in comparison with r, h^2 is smaller still and may be neglected;

therefore practically $\quad PT^2 = 2rh,$

or $\quad \boxed{PT = \sqrt{2rh}.}$

Also $\quad \tan TPF = \tan TCP = \dfrac{PT}{TC}$

$$= \frac{\sqrt{2rh}}{r}$$

$$= \sqrt{\frac{2h}{r}},$$

and since TPF is a small angle,

$$\therefore \; T\hat{P}F = \tan TPF = \sqrt{\frac{2h}{r}} \text{ radians}$$

$$\boxed{= \sqrt{\frac{2h}{r}} \cdot \frac{180^\circ}{\pi} \cdot}$$

Ex. 1. Find the distance and dip of the horizon from the top of the mast of a ship 120 feet above sea-level, assuming that the radius of the earth is 4000 miles.

Since 120 ft. is very small in comparison with 4000 miles,

$$\therefore \text{ distance of horizon} = x = \sqrt{2rh}$$

$$= \sqrt{\frac{2 \times 4000 \times 120}{1760 \times 3}} \text{ miles}$$

$$= \sqrt{\frac{2000}{11}} \text{ miles.}$$

$$\therefore \log x = \tfrac{1}{2}(\log 2000 - \log 11)$$

$$\log 2000 = 3{\cdot}3010$$

$$\log 11 = 1{\cdot}0414$$

$$2 \,|\, 2{\cdot}2596$$

$$\therefore \log x = 1{\cdot}1298$$

$$\therefore x = 13{\cdot}48 \text{ miles.}$$

$$\text{Dip of Horizon} = D = \sqrt{\frac{2h}{r}} \cdot \frac{180^\circ}{\pi}$$

$$= \sqrt{\frac{2 \times 120}{1760 \times 3 \times 4000}} \cdot \frac{180 \times 7 \times 60'}{22}$$

$$= \sqrt{\frac{1}{22 \times 4000}} \cdot \frac{37800'}{11}$$

$\log 37800 = 4{\cdot}5775$

$3{\cdot}5137$

$\therefore \log D = 1{\cdot}0638$

$\therefore D = 11{\cdot}58'$

$\log 22 = 1{\cdot}3424$

$\log 4000 = 3{\cdot}6021$

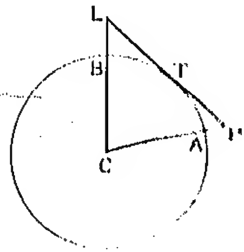
$2 \,|\, 4{\cdot}9445$

$2{\cdot}4723$

$\log 11 = 1{\cdot}0414$

$3{\cdot}5137.$

Ex. 2. From the top of the mast of a ship 70 feet above sea-level, the light from a lighthouse 145 feet high can just be seen. If the radius of the earth is assumed to be 4000 miles, what is the distance between the ship and lighthouse?



$$\text{PA} = 70 \text{ feet},$$

$$\text{LB} = 145 \text{ feet},$$

$$\text{arc AT} = \text{PT (approx.)} = \sqrt{2rh}$$

$$= \sqrt{\frac{2 \times 4000 \times 70}{1760 \times 3}} \text{ miles}$$

$$= \sqrt{\frac{3500}{33}} \text{ miles},$$

[illegible]

[illegible] in comp[illegible]

$$\therefore \log \text{AT} = 1{\cdot}0128 \qquad \therefore \text{arc AT} = 10{\cdot}30 \text{ miles}.$$

$$\text{arc TB} = \text{TL (approx.)}$$

$$= \sqrt{\frac{2 \times 4000 \times 145}{1760 \times 3}} \text{ miles}$$

$$= \sqrt{\frac{7250}{33}} \text{ miles},$$

$$\log 7250 = 3{\cdot}8603$$

$$\log 33 = 1{\cdot}5185$$

$$2 \,|\, 2{\cdot}3418$$

$$\therefore \log \text{TB} = 1{\cdot}1709 \qquad \therefore \text{arc TB} = 14{\cdot}82 \text{ miles}.$$

$$\therefore \text{Distance AB} = \text{AT} + \text{TB} = 25{\cdot}12 \text{ miles}.$$

EXAMPLES XXVII.

[$\pi = \frac{22}{7}$, radius of earth $= 4000$ miles.]

1. Find the distance of the visible horizon from the top of a mound 350 feet high.

2. Find the dip of the horizon from the top of a lighthouse 1[illegible]6 feet high.

3. The lamp of a lighthouse is 200 feet high; how far away can it be seen?

4. From the top of one lighthouse 150 feet high the light of another 200 feet high can just be seen. Calculate the approximate distance apart of the lighthouses.

5. From the top of a ship's mast 70 feet above sea-level, the lamp of a lighthouse can just be seen. After sailing towards the lighthouse for 1 hour the lamp can be seen from the deck, 20 feet above sea-level. Find the rate of the ship's sailing.

6. From the mast of a ship 80 feet high the lamp of a lighthouse is just visible at a distance of 30 miles. What is the height of the lamp?

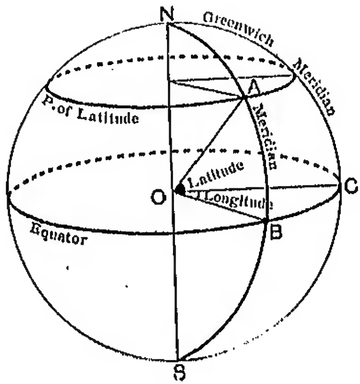
Miscellaneous Examples on Chapter X start in Test Paper XXXV, p. 228.

A *Meridian* is a great circle passing through the terrestrial poles (NABS).

A small circle of the earth parallel to the equator is called a *Parallel of Latitude.*

The *Latitude* of a place is its angular distance from the equator measured along the meridian ($A\hat{O}B$).

The *Longitude* of a place is the angle between the meridian through that place and a certain fixed meridian (usually that of Greenwich) ($C\hat{O}B$).



EXAMPLES XXVII A.

($\pi = \frac{22}{7}$.)

1. Two places are on the equator and 200 miles apart. If the earth's radius is 4000 miles, find their difference in longitude.

2. Two places on the same meridian have latitudes 25° N. and 32° S. Find their distance apart measured on the earth's surface. (Radius = 4000 miles.)

3. The latitude of Edinburgh is 56° N. Find its distance from the earth's axis and also its distance, measured along the surface, from the equator. (Radius = 4000 miles.)

4. The arc of the meridian of longitude between Chartres in latitude 48° 25′ and Toulouse in latitude 43° 35′ is 335 miles. Find the radius of the earth to the nearest tenth of a mile.

5. Two places on the same meridian are 287·2 miles apart. If the earth's diameter is 7920 miles, find their difference in latitude to the nearest minute.

6. Find the difference in latitude of two places on the same meridian on a globe of diameter 6 feet, if their distance apart, measured along the surface of the globe, is 17 inches. (Answer to the nearest minute.)

7. Two places A and B on the earth's surface are on the same parallel of latitude 52° 30′. The difference of their longitudes is 32° 15′. Take the earth a sphere of such a size that a mile on the surface subtends an angle of 1′ at the centre, and find

(i) the radius of the parallel of latitude on which A and B lie,

(ii) the distance in a straight line between A and B,

(iii) the distance between A and B along a great circle, that is, along a circle which passes through these points and has its centre at the centre of the earth.

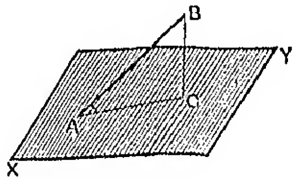
$$(\pi = 3{\cdot}1416.)$$

8. The ancient Greeks measured the latitude of a place by setting up a vertical rod and comparing its length with the length of its shadow. Supposing observations taken at mid-day at the equinox (when the sun is vertical at the equator) to give $\frac{8}{3}$ as the ratio of rod to shadow at Alexandria, and $\frac{4}{3}$ as the ratio at Carthage, find the latitude of each place.

9. A ship sails at 7 miles an hour along the parallel of latitude 45° from Halifax in longitude 63° 40′ W. to Bordeaux in longitude 20′ W. If the radius of the earth is 4200 miles, what is the time of the voyage?

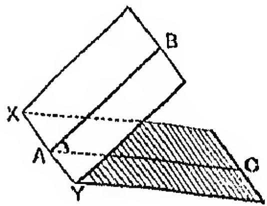
10. A and B are two places on the earth's surface in latitude 60°, and their difference of longitude is 32°. If the earth be taken as a sphere such that a mile measured along the equator subtends an angle 1′ at the centre, find the distance between A and B measured along the parallel of latitude. Taking the earth's radius as R, obtain an expression for the length of the chord AB.

The angle between a straight line and a plane is the angle between that line and its projection on the plane.



If BC is drawn perp. to the plane XY, then the angle between AB and the plane XY is BAC.

The angle between two planes is the angle between two straight lines drawn perp. to the line of common section, one in each plane.

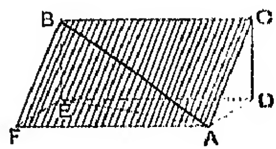


If XY is the line of section and AB, AC are each perp. to XY, then BAC is the angle between the planes.

It is proved in books on Geometry, that if a straight line is perp. to two intersecting straight lines it is perp. to the plane which contains them.

Ex. 1. *On a hill sloping at* 18° *to the horizontal plane, runs a track making an angle of* 50° *with the line of greatest slope. What is the length of the track to the top of the hill which is* 1500 *feet high, and what angle does the track make with the horizontal plane?*

Let AB be the track, FACB the hill side, rectangular in shape, and ADEF the horizontal plane through FA, E and D being the projections of B and C.



$$D\hat{A}C = 18°, \quad C\hat{A}B = 50°,$$

$$EB = DC = 1500 \text{ ft.}$$

$$\frac{AB}{1500} = \frac{AB}{AC} \cdot \frac{AC}{CD} = \frac{1}{\cos 50° \sin 18°},$$

$$\therefore AB = \frac{1500}{\cos 50° \sin 18°}.$$

$\log 1500 =$	$3{\cdot}1761$	$L \sin 18° =$	$9{\cdot}4900$
	$19{\cdot}2981$	$L \cos 50° =$	$9{\cdot}8081$
$\therefore \log AB =$	$3{\cdot}8780$		$19{\cdot}2981$

$\therefore AB = 7551$ feet.

Since AE is the projection of AB on the horizontal plane,

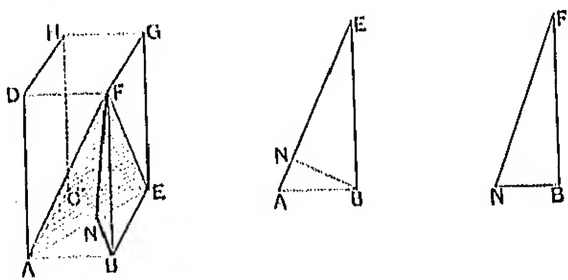
$\therefore$ angle AB makes with horizontal is BAE.

$$\sin BAE = \frac{BE}{BA} = \frac{1500 \sin 18° \cos 50°}{1500} = \sin 18° \cos 50°,$$

$$\therefore L \sin BAE = 9{\cdot}2981,$$

$$\therefore B\hat{A}E = 11° \, 28'.$$

Ex. 2. *If the three coterminous edges* AB, AC, AD *of a rectangular solid are* 1, 2, 3 *inches respectively, find the angle between the plane* AFE *and the base* ABEC.



Draw FN perp. to AE, then since NB is the projection of NF on the plane ABEC, it follows that the angle between AFE and ABEC is FNB.

$$BN \,.\, AE = 2 \triangle ABE = AB \,.\, BE,$$

$$\therefore BN = \frac{1 \times 2}{\sqrt{5}} = \frac{2}{\sqrt{5}},$$

$$\tan BNF = \frac{BF}{NB} = \frac{3\sqrt{5}}{2} = 3{\cdot}3540,$$

$$\therefore B\hat{N}F = 73° \, 23'.$$

EXAMPLES XXVIIb.

1. In a rectangular solid, three coterminous edges AB, AC, AD are 2, 4, 5 inches respectively; find the angles between the diagonal AE of the solid and these three edges.

2. A door ABCD, 7 ft. by 3 ft. 6 in., is turned round the line of hinges AD through an angle of 25° into a new position AB′C′D; calculate the length of CC′ and the angle CAC′.

3. A right-angled triangle ABC, right-angled at B, in which $AB = 3$, $BC = 4$, is in a horizontal plane and is rotated round AC through 24° into the position ACB′. Calculate the angles which B′A and B′C make with the horizontal.

4. A set-square ABC, right-angled at B, is in a vertical plane, $AB = 4$ and $BC = 5$ inches, AB being vertical and BC horizontal. It is rotated round AB through an angle of 38° into the position ABC′. Find the angle CAC′.

Verify by a figure drawn to scale. [If D is the middle pt. of CC′, draw $\triangle^s$ CBC′, ABD, ADC.]

5. In any cube, find the angle between a diagonal and the diagonal of a face which meets it; also find the angle between the diagonals of any two adjacent faces.

6. OABCD is a right pyramid on a square base ABCD, the height being 12 cm. and the side of the base 8 cm. Find the angles between

(i) OB and OD,

(ii) OB and BC,

(iii) OB and the plane ABCD,

(iv) the planes OAB and ABCD,

(v) the planes OAB and OBC.

7. If ABCD, PQRS are the floor and ceiling of a room, A being vertically above P, B above Q, etc., calculate the angles between AR and AC, and between AR and AS.

AB = 18 feet, BC = 16 feet, AP = 13 feet.

8. The plane face of a hill slopes at 10° to the horizontal; a path 50 yards long on the hill makes an angle of 50° with the line of greatest slope; calculate the vertical height of the top of the path above the bottom.

9. A drawing board ABCD slopes at an angle of 35° to the horizontal; the lower side BC is horizontal, and O a point in it; OP is 10 inches long, lies on the drawing board, and makes an angle of 40° with BC. Find the vertical height of P above BC and deduce the slope of OP.

10. A chasm in level ground is bounded by parallel vertical sides. The depth AB of the chasm at A is wanted, and, it being impossible to take measurements from C, the point opposite A, a point D, 50 yards along the side from C, is chosen. The angle ADB is 43° and the angle ADC is 52°. Find the depth AB.

11. A regular pyramid has a square base. The faces are isosceles triangles having the base angles equal to 70°. Suppose such a pyramid made of cardboard, the base measuring 4 cm. in the side. Suppose it slit along the slant sides and spread out flat, and draw this flat figure.

Find the inclination of the faces of the original pyramid to the base by drawing and by calculation.

12. The line A of greatest slope up the plane face of a hill rises 28 in 100 (28 vertical and 100 horizontal). Determine graphically the slope of a path B which makes with A an angle of 40°. Give the answer as so many in 100.

CHAPTER XI.

FUNCTIONS OF COMPOUND ANGLES.

Important formulae proved in this chapter:

1. $\sin(A+B)=\sin A\cos B+\cos A\sin B.$
2. $\sin(A-B)=\sin A\cos B-\cos A\sin B.$
3. $\cos(A+B)=\cos A\cos B-\sin A\sin B.$
4. $\cos(A-B)=\cos A\cos B+\sin A\sin B.$
5. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}.$
6. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}.$
7. $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$
8. $\sin 2A=2\sin A\cos A.$
9. $\cos 2A=\cos^2 A-\sin^2 A.$
10. $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}.$
11. $\sin 3A=3\sin A-4\sin^3 A.$
12. $\cos 3A=4\cos^3 A-3\cos A.$
13. $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}.$

105. To prove Geometrically the Formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B.$$

Case (i) when $A+B < 90^\circ$.

Let $N\hat{O}M = A, \quad N\hat{O}P = B,$

then $P\hat{O}L = A+B.$

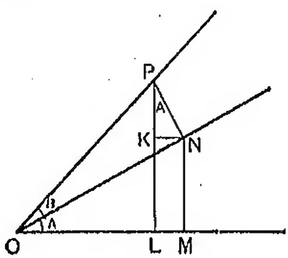
Let $OP =$ unit of length.

Draw PN, NM, PL, NK perpendicular to ON, OM, OM, PL respectively.

Then since PN and PK are respectively perpendicular to ON and OM,

$$\therefore\ K\hat{P}N = M\hat{O}N = A.$$

Also
$$ON = OP\cos B = \cos B$$
$$PN = OP\sin B = \sin B$$
$$LM = KN = PN\sin A = \sin B\sin A$$
$$OM = ON\cos A = \cos B\cos A$$
$$OL = OP\cos(A+B) = \cos(A+B).$$



Now
$$OL = OM - LM;$$
$$\therefore\ \cos(A+B) = \cos A\cos B - \sin A\sin B.$$

Also
$$PK = PN\cos A = \sin B\cos A$$
$$KL = NM = ON\sin A = \cos B\sin A$$
$$PL = OP\sin(A+B) = \sin(A+B).$$

Now
$$PL = KL + PK;$$
$$\therefore\ \sin(A+B) = \sin A\cos B + \sin B\cos A.$$

106. ALITER.

$$\sin(A+B) = \frac{PL}{OP} = \frac{KL+PK}{OP} = \frac{NM}{OP} + \frac{PK}{OP}$$

$$= \frac{NM}{ON} \cdot \frac{ON}{OP} + \frac{PK}{PN} \cdot \frac{PN}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

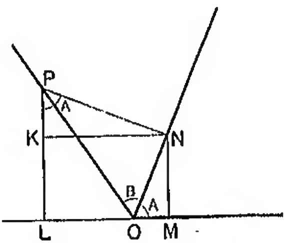
$$\cos(A+B) = \frac{OL}{OP} = \frac{OM-LM}{OP}$$

$$= \frac{OM}{OP} - \frac{KN}{OP}$$

$$= \frac{OM}{ON} \cdot \frac{ON}{OP} - \frac{KN}{PN} \cdot \frac{PN}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

107. Case (ii) when A and B are both acute, but $A+B > 90°$.



Construction as before:

$$ON = OP \cos B = \cos B$$

$$PN = OP \sin B = \sin B$$

$$LM = KN = PN \sin A = \sin B \sin A$$

$$OM = ON \cos A = \cos B \cos A$$

$$OL = OP \cos P\hat{O}L = -\cos(A+B).$$

Now $\qquad -OL = OM - LM;$

$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Also $\qquad PK = PN \cos A = \sin B \cos A$

$$KL = NM = ON \sin A = \cos B \sin A$$

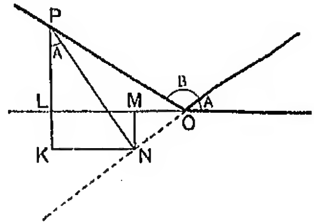
$$PL = OP \sin P\hat{O}L = \sin(A+B).$$

Now $\qquad PL = KL + PK;$

$$\therefore \sin(A+B) = \sin A \cos B + \sin B \cos A.$$

108. Case (iii) A acute, B obtuse but $A+B<180°$.

Construction as before:



$ON = OP\cos P\hat{O}N = -\cos B$

$PN = OP\sin P\hat{O}N = \sin B$

$LM = KN = PN\sin A = \sin B\sin A$

$OM = ON\cos A = -\cos B\cos A$

$OL = OP\cos P\hat{O}L = -\cos(A+B).$

Now

$$OL = OM + LM,$$

$$\therefore\ -\cos(A+B) = -\cos B\cos A + \sin B\sin A;$$

$$\therefore\ \cos(A+B) = \cos A\cos B - \sin A\sin B.$$

Also $\quad PK = PN\cos A = \sin B\cos A$

$KL = MN = ON\sin A = -\cos B\sin A$

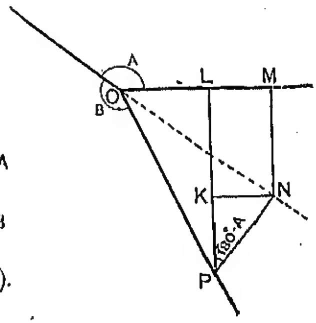
$PL = OP\sin(A+B) = \sin(A+B).$

Now $\quad PL = PK - KL;$

$$\therefore\ \sin(A+B) = \sin B\cos A + \cos B\sin A.$$

109. Case (iv) when A and B are both obtuse, but $A+B>270°$.

Construction as before:



$ON = OP\cos P\hat{O}N = -\cos B$

$PN = OP\sin PON = \sin B$

$LM = KN = PN\sin(180° - A)$
$= \sin B\sin A$

$OM = ON\cos M\hat{O}N$
$= (-\cos B)(-\cos A) = \cos A\cos B$

$OL = OP\cos POL$
$= \cos\{360° - (A+B)\} = \cos(A+B).$

Now $\quad OL = OM - LM;$

$$\therefore\ \cos(A+B) = \cos A\cos B - \sin B\sin A.$$

Also $\quad PK = PN\cos(180° - A) = -\sin B\cos A$

$KL = NM = ON\sin(180° - A) = -\cos B\sin A$

$PL = OP\sin\{360° - (A+B)\} = -\sin(A+B).$

Now $\quad PL = KL + PK;$

$$\therefore\ \sin(A+B) = \sin A\cos B + \cos A\sin B.$$

There are other cases which are left as an exercise for the student.

110. To prove geometrically the Formulae

$$\sin(A-B) = \sin A\cos B - \sin B\cos A,$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Case (i) when A and B < 90°.

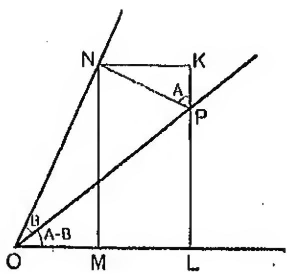
Let $\quad N\hat{O}M = A, \quad N\hat{O}P = B,$

then $\quad P\hat{O}L = A - B.$

Let $\quad OP = \text{unit of length}.$

Draw PN, NM, PL, NK perpendicular to ON, OM, OL, PL respectively.

Then since PN and PK are respectively perpendicular to ON and OM



$$\therefore\quad K\hat{P}N = M\hat{O}N = A,$$

also
$$ON = OP\cos B = \cos B$$
$$PN = OP\sin B = \sin B$$
$$LM = KN = PN\sin A = \sin B\sin A$$
$$OM = ON\cos A \qquad = \cos B\cos A$$
$$OL = OP\cos(A-B) = \cos(A-B).$$

Now
$$OL = OM + LM;$$
$$\therefore\quad \cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Also
$$PK = PN\cos A = \sin B\cos A$$
$$KL = NM = ON\sin A = \cos B\sin A$$
$$PL = OP\sin(A-B) = \sin(A-B).$$

Now
$$PL = KL - KP;$$
$$\therefore\quad \sin(A-B) = \sin A\cos B - \sin B\cos A.$$

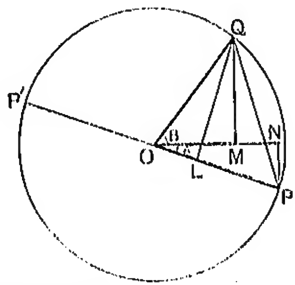
The other cases where A and B are larger angles are left for the student, they are similar to those on pp. 162 and 163.

111. ALITER.

$$\sin(A-B)=\frac{PL}{OP}=\frac{KL-PK}{OP}$$
$$=\frac{NM}{OP}-\frac{PK}{OP}$$
$$=\frac{NM}{ON}\cdot\frac{ON}{OP}-\frac{PK}{PN}\cdot\frac{PN}{OP}$$
$$=\sin A\cos B-\cos A\sin B.$$

$$\cos(A-B)=\frac{OL}{OP}=\frac{OM+ML}{OP}$$
$$=\frac{OM}{OP}+\frac{ML}{OP}$$
$$=\frac{OM}{ON}\cdot\frac{ON}{OP}+\frac{KN}{NP}\cdot\frac{NP}{OP}$$
$$=\cos A\cos B+\sin A\sin B.$$

112. The student will find the following geometrical proof interesting.



The figure explains itself.

Let the radius of the circle = the unit of length.

Then $$PQ^2=(PN+QM)^2+MN^2=(PN+QM)^2+(ON-OM)^2$$
$$=(\sin A+\sin B)^2+(\cos A-\cos B)^2$$
$$=2+2\sin A\sin B-2\cos A\cos B.$$

Again, $$PQ^2=PL\,.\,PP'=2PL$$
$$=2(OP-OL)$$
$$=2-2OQ\cos(A+B)$$
$$=2-2\cos(A+B),$$

$$\therefore\ 2-2\cos(A+B)=2+2\sin A\sin B-2\cos A\cos B;$$
$$\therefore\ \cos(A+B)=\cos A\cos B-\sin A\sin B.$$

ILLUSTRATIVE EXAMPLES.

Ex. 1. To find the value of sin 105°.

$$\sin 105° = \sin(60° + 45°)$$
$$= \sin 60° \cos 45° + \sin 45° \cos 60°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Ex. 2. To find the value of cos 15°.

$$\cos 15° = \cos(45° - 30°)$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

The student will notice that we might have said

$$\sin 105° = \sin(90° + 15°) = \cos 15°.$$

Ex. 3. To prove $\cos(90° + A) = -\sin A$.

$$\cos(90° + A) = \cos 90° \cos A - \sin 90° \sin A$$
$$= 0 - \sin A$$
$$= -\sin A.$$

Ex. 4. Prove that

$$\cos 68° 20' \cos 8° 20' + \cos 81° 40' \cos 21° 40' = \tfrac{1}{2}.$$

We notice that 68° 20′ and 21° 40′ are complementary, and also that 8° 20′ and 81° 40′ are complementary,

and remembering sin (any angle) = cos (its complement),

$$\cos 68° 20' \cos 8° 20' + \cos 81° 40' \cos 21° 40'$$
$$= \sin 21° 40' \cos 8° 20' + \sin 8° 20' \cos 21° 40'$$
$$= \sin(21° 40' + 8° 20')$$
$$= \sin 30°$$
$$= \tfrac{1}{2}.$$

EXAMPLES XXVIII.

[Pr]ove that

1. $\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

2. $\cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}}$.

3. $\sin (90^\circ - A) = \cos A$.

4. $\sin (90^\circ + A) = \cos A$.

5. $\sin (180^\circ - A) = \sin A$.

6. $\sin (180^\circ + A) = -\sin A$.

7. $\cos (90^\circ - A) = \sin A$.

8. $\cos (90^\circ + A) = -\sin A$.

9. $\cos (180^\circ - A) = -\cos A$.

10. $\cos (180^\circ + A) = -\cos A$.

11. $\sin 23^\circ \cos 7^\circ + \cos 23^\circ \sin 7^\circ = \frac{1}{2}$.

12. $\cos 83^\circ \cos 23^\circ + \sin 83^\circ \sin 23^\circ = \frac{1}{2}$.

13. $\sin 78^\circ 32' \cos 11^\circ 28' + \sin 11^\circ 28' \cos 78^\circ 32' = 1$.

14. $\sin 17^\circ 26' \cos 12^\circ 34' + \sin 72^\circ 34' \sin 12^\circ 34' = \frac{1}{2}$.

15. $\cos (A + \theta) \cos (A - \theta) - \sin (A + \theta) \sin (A - \theta)$ is independent [o]f θ.

16. $\sin (X + 40^\circ) \cos (X + 30^\circ) - \cos (X + 40^\circ) \sin (X + 30^\circ)$ is [i]ndependent of X.

17. $\sin (A + 45^\circ) = \frac{1}{\sqrt{2}} (\sin A + \cos A)$.

18. $\sin (A - 45^\circ) = \frac{1}{\sqrt{2}} (\sin A - \cos A)$.

19. $\cos (A + 45^\circ) = \frac{1}{\sqrt{2}} (\cos A - \sin A)$.

20. $\cos (A - 45^\circ) = \frac{1}{\sqrt{2}} (\cos A + \sin A)$.

21. $\sin(A+30^\circ)=\frac{1}{2}(\sqrt{3}\sin A+\cos A)$.

22. $\cos(A-30^\circ)=\frac{1}{2}(\sqrt{3}\cos A+\sin A)$.

23. $\sin(A+B)+\cos(A-B)=(\sin A+\cos A)(\sin B+\cos B)$.

24. $\sin(A-B)+\cos(A+B)=(\sin A+\cos A)(\cos B-\sin B)$.

25. $2\sin\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}+B\right)=\cos(A+B)+\sin(A-B)$.

26. $2\cos\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}-B\right)=\cos(A+B)-\sin(A-B)$.

27. Given $\sin A=\frac{\sqrt{3}}{2}$, $\cos B=\frac{1}{\sqrt{2}}$ and that A and B are acute, find $\sin(A+B)$ and $\cos(A-B)$.

28. $\sin A=\frac{1}{2}$, $\cos B=\frac{1}{\sqrt{2}}$, find $\sin(A+B)$ and $\cos(A-B)$.

29. $\sin A=\frac{1}{\sqrt{2}}$, $\cos B=\frac{\sqrt{3}}{2}$, find $\sin(A-B)$ and $\cos(A+B)$.

Prove that

30. $\cos 5A\cos 3A+\sin 5A\sin 3A=\cos 2A$.

31. $\sin 2A\cos A+\sin A\cos 2A=\sin 3A$.

The following are important formulae:

113. (i) $\tan(A+B)$

$$=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\sin B\cos A}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\sin B\cos A}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

(ii) $\tan(A-B)$

$$=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\sin B\cos A}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\sin B\cos A}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}+\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

114. (i) $\cot(A+B)$

$$\frac{\cos(A+B)}{\sin(A+B)}=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-\dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B}+\dfrac{\sin B\cos A}{\sin A\sin B}}$$

$$=\frac{\cot A\cot B-1}{\cot A+\cot B}.$$

(ii) $$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

115. (i) $\sin(A+B+C)$

$=\sin A\cos(B+C)+\cos A\sin(B+C)$

$=\sin A\cos B\cos C-\sin A\sin B\sin C$

$\quad+\cos A\sin B\cos C+\cos A\sin C\cos B$

$=\Sigma\sin A\cos B\cos C-\sin A\sin B\sin C$

$=\cos A\cos B\cos C(\tan A+\tan B+\tan C$

$\quad-\tan A\tan B\tan C).$

(ii) $\cos(A+B+C)$

$=\cos A\cos(B+C)-\sin A\sin(B+C)$

$=\cos A\cos B\cos C-\cos A\sin B\sin C$

$\quad-\sin A\sin B\cos C-\sin A\sin C\cos B$

$=\cos A\cos B\cos C-\Sigma\sin A\sin B\cos C$

$=\cos A\cos B\cos C(1-\tan B\tan C$

$\quad-\tan C\tan A-\tan A\tan B).$

(iii) $\tan(A+B+C) = \dfrac{\sin(A+B+C)}{\cos(A+B+C)}$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

116. $\sin(A+B)\sin(A-B)$

$$= (\sin A \cos B + \sin B \cos A)(\sin A \cos B - \sin B \cos A)$$

$$= \sin^2 A \cos^2 B - \sin^2 B \cos^2 A$$

$$= \sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)$$

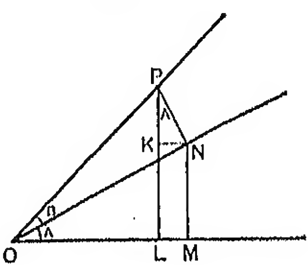
$$= \sin^2 A - \sin^2 B.$$

117. The tangent formulae may be proved geometrically.

$$\tan(A+B) = \frac{PL}{OL} = \frac{PK + NM}{OM - LM} = \frac{\dfrac{PK}{OM} + \dfrac{NM}{OM}}{1 - \dfrac{KN}{OM}}$$

$$= \frac{\dfrac{PK}{OM} + \dfrac{NM}{OM}}{1 - \dfrac{KN}{PK}\cdot\dfrac{PK}{OM}}$$

$$= \frac{\dfrac{PN}{ON} + \dfrac{NM}{OM}}{1 - \dfrac{KN}{PK}\cdot\dfrac{PN}{ON}}$$



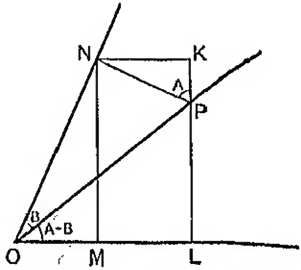
(since triangles KPN and MON are similar)

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

118. $\tan(A-B)=\dfrac{PL}{OL}=\dfrac{NM-PK}{OM+NK}$

$$=\frac{\dfrac{NM}{OM}-\dfrac{PK}{OM}}{1+\dfrac{NK}{PK}\cdot\dfrac{PK}{OM}}$$

$$=\frac{\dfrac{NM}{OM}-\dfrac{PN}{ON}}{1+\dfrac{NK}{PK}\cdot\dfrac{PN}{ON}}$$



(since triangles MON and KPN are similar)

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

EXAMPLES XXIX.

Prove that

1. $\tan 75^\circ=\cot 15^\circ=2+\sqrt{3}$.
2. $\tan 15^\circ=\cot 75^\circ=2-\sqrt{3}$.
3. $\tan(90^\circ-A)=\cot A$.
4. $\tan(90^\circ+A)=-\cot A$.
5. $\tan(180^\circ-A)=-\tan A$.
6. $\tan(180^\circ+A)=\tan A$.
7. $\tan(A+45^\circ)=\dfrac{\tan A+1}{1-\tan A}$.
8. $\tan(A-45^\circ)=\dfrac{\tan A-1}{1+\tan A}$.
9. $\tan(A+30^\circ)=\dfrac{\sqrt{3}\tan A+1}{\sqrt{3}-\tan A}$.

10. $\tan(A+60^\circ)=\dfrac{\tan A+\sqrt{3}}{1-\sqrt{3}\tan A}$.

11. $\tan 15^\circ+\tan 30^\circ+\tan 15^\circ\tan 30^\circ=1$.

12. $\tan 75^\circ-\tan 30^\circ-\tan 75^\circ\tan 30^\circ=1$.

Expand

13. $\sin(A+B+C)$.

14. $\cos(A+B+C)$.

15. $\tan(A+B+C)$.

16. $\sin(A+B-C)$.

17. $\cos(A-B+C)$.

18. $\tan(A-B-C)$.

Prove that

19. $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$
$=\cos^2 B-\cos^2 A$.

20. $\cos(A+B)\cos(A-B)=\cos^2 B-\sin^2 A$
$=\cos^2 A-\sin^2 B$.

Important formulae deduced from the four fundamental formulae.

119. Since $\sin(A+B)=\sin A\cos B+\sin B\cos A$,

therefore putting $B=A$,

we obtain $\sin 2A=\sin A\cos A+\sin A\cos A$

$$=2\sin A\cos A.$$

This formula is perfectly general and we may therefore say

$$\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

$$\sin 24^\circ=2\sin 12^\circ\cos 12^\circ, \text{ etc.}$$

120. $\cos(A+B)=\cos A\cos B-\sin A\sin B.$

Put $$B=A,$$

then $$\cos 2A=\cos^2 A-\sin^2 A,$$

$$=2\cos^2 A-1, \qquad \text{since } \sin^2 A=1-\cos^2 A$$

$$=1-2\sin^2 A, \qquad \text{since } \cos^2 A=1-\sin^2 A,$$

or $$1+\cos 2A=2\cos^2 A$$

$$1-\cos 2A=2\sin^2 A.$$

$$\therefore\quad \tan^2 A=\frac{1-\cos 2A}{1+\cos 2A}.$$

These formulae being perfectly general we may say

$$\cos\theta=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$$

$$=2\cos^2\frac{\theta}{2}-1$$

$$=1-2\sin^2\frac{\theta}{2},$$

$$\cos 30^\circ=\cos^2 15^\circ-\sin^2 15^\circ, \text{ etc.}$$

121. $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Put $$B=A,$$

then $$\tan 2A=\frac{2\tan A}{1-\tan^2 A}.$$

This formula being perfectly general we may say

$$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},$$

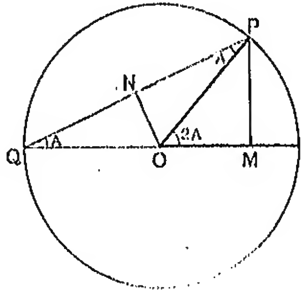
$$\tan 45^\circ=\frac{2\tan 22\tfrac{1}{2}^\circ}{1-\tan^2 22\tfrac{1}{2}^\circ}, \text{ etc.}$$

122. $$\sin(A+2A) = \sin A \cos 2A + \sin 2A \cos A$$
$$= \sin A(1 - 2\sin^2 A) + 2\sin A \cos^2 A$$
$$= \sin A(1-2\sin^2 A) + 2\sin A(1-\sin^2 A).$$
$$\sin 3A = 3\sin A - 4\sin^3 A.$$

123. $$\cos(A+2A) = \cos A \cos 2A - \sin A \sin 2A$$
$$= \cos A(2\cos^2 A - 1) - 2\sin^2 A \cos A$$
$$= \cos A(2\cos^2 A - 1) - 2(1-\cos^2 A)\cos A.$$
$$\cos 3A = 4\cos^3 A - 3\cos A.$$

124. $$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A},$$
$$\therefore \tan 3A = \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \dfrac{2\tan^2 A}{1-\tan^2 A}}$$
$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

125. To prove geometrically the formulae for sin 2A and cos 2A.



The figure explains itself.

Let the radius be the unit of length.

$$PQ = 2QN = 2OQ \cos A = 2 \cos A.$$

$$PM = OP \sin 2A = \sin 2A,$$

also
$$PM = PQ \sin A = 2 \cos A \sin A,$$

$$\therefore \sin 2A = 2 \sin A \cos A.$$

Again
$$OM = OP \cos 2A = \cos 2A$$

and
$$\begin{aligned} OM &= QM - OQ \\ &= QP \cos A - 1 \\ &= 2 \cos^2 A - 1. \end{aligned}$$

$$\therefore \cos 2A = 2 \cos^2 A - 1.$$

126. To find the values of sin 3A and cos 3A geometrically.

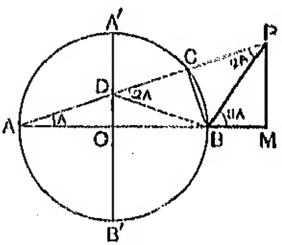
Take a circle with AB and A′B′ as perpendicular diameters.

Make $B\hat{A}C = A$ and produce AC to P so that $DC = CP$.

Then $PB = DB = AD$

$$C\hat{D}B = D\hat{A}B + D\hat{B}A = 2A = C\hat{P}B$$

$$P\hat{B}M = B\hat{A}P + B\hat{P}A = 3A.$$



$$\therefore \sin 3A = \frac{PM}{PB} = \frac{AP \sin A}{PB}$$

$$= \frac{2\,AC - AD}{PB} \cdot \sin A$$

$$= \left(2\frac{AC}{AD} - 1\right) \sin A.$$

Now since triangles OAD and CAB are similar

$$\frac{AC}{AB} = \frac{AO}{AD};$$

$$\therefore AC = \frac{2AO^2}{AD},$$

$$\therefore \frac{AC}{AD} = \frac{2AO^2}{AD^2} = 2 \cos^2 A.$$

$$\therefore \quad \sin 3A = (4\cos^2 A - 1)\sin A$$
$$= (3 - 4\sin^2 A)\sin A.$$

$$\cos 3A = \frac{BM}{PB}$$
$$= \frac{AM - AB}{AD}$$
$$= \frac{AP\ \cos\ A}{AD} - \frac{2AO}{AD}$$
$$= (4\cos^2 A - 1)\cos A - 2\cos A, \text{ as above}$$
$$= 4\cos^3 A - 3\cos A.$$

ILLUSTRATIVE EXAMPLES.

Ex. 1. Find the value of sin 18°.

Notice $5 \times 18° = 90°$,

$$\therefore \quad 2 \times 18° = 90° - 3 \times 18°,$$
$$\therefore \quad \sin(2 \times 18°) = \cos(3 \times 18°),$$
$$\therefore \quad 2\sin 18° \cos 18° = 4\cos^3 18° - 3\cos 18°,$$
$$\therefore \quad 2\sin 18° = 4\cos^2 18° - 3,$$
since $\cos 18° \neq 0$,
$$= 4(1 - \sin^2 18°) - 3,$$
$$\therefore \quad 4\sin^2 18° + 2\sin 18° - 1 = 0,$$
$$\therefore \quad \sin 18° = \frac{-1 \pm \sqrt{5}}{4}$$
$$= \frac{\sqrt{5} - 1}{4},$$
since sin 18° = a positive quantity.

Ex. 2. To show

$$\cos^8 A - \sin^8 A = \tfrac{1}{2}\cos 2A + \tfrac{1}{2}\cos^3 2A.$$

$$\begin{aligned}\cos^8 A - \sin^8 A &= (\cos^4 A - \sin^4 A)\{\cos^4 A + \sin^4 A\}\\ &= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)\\ &\qquad\{(\cos^2 A + \sin^2 A)^2 - 2\sin^2 A\cos^2 A\}\\ &= \cos 2A\{1 - \tfrac{1}{2}\sin^2 2A\}\\ &= \cos 2A\{1 - \tfrac{1}{2}(1 - \cos^2 2A)\}\\ &= \tfrac{1}{2}\cos 2A + \tfrac{1}{2}\cos^3 2A.\end{aligned}$$

Ex. 3. Express $\cos 6A$ in terms of $\cos A$.

$$\begin{aligned}\cos 6A &= \cos(3A + 3A)\\ &= \cos^2 3A - \sin^2 3A\\ &= 2\cos^2 3A - 1\\ &= 2(4\cos^3 A - 3\cos A)^2 - 1\\ &= 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1.\end{aligned}$$

Ex. 4. To prove $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A.$

$$\begin{aligned}\sec 2A + \tan 2A &= \frac{1 + \sin 2A}{\cos 2A}\\ &= \frac{1 + 2\sin A\cos A}{\cos^2 A - \sin^2 A}\\ &= \frac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A}\\ &= \frac{(\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}\\ &= \frac{\cos A + \sin A}{\cos A - \sin A}.\end{aligned}$$

EXAMPLES XXX.

Prove that

1. $\cos 18^\circ = \sin 72^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

2. $\sin 36^\circ = \cos 54^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$.

3. $\cos 36^\circ = \sin 54^\circ = \dfrac{\sqrt{5}+1}{4}$.

Find $\sin 2A$, $\cos 2A$, $\tan 2A$, when A is acute and

4. (i) $\sin A = \frac{1}{2}$, (ii) $\sin A = \frac{1}{5}$,
(iii) $= \frac{1}{4}$, (iv) $= \frac{3}{5}$,
(v) $= \frac{12}{13}$.

5. (i) $\cos A = \dfrac{\sqrt{3}}{2}$, (ii) $\cos A = \dfrac{1}{\sqrt{2}}$,
(iii) $= \frac{3}{5}$, (iv) $= \frac{12}{13}$,
(v) $= \frac{1}{5}$.

6. (i) $\tan A = \sqrt{3}$, (ii) $\tan A = 1$,
(iii) $= \frac{3}{4}$, (iv) $= \frac{5}{12}$,
(v) $= \frac{2}{7}$, (vi) $= \frac{3}{11}$.

7. Find $\tan \dfrac{A}{2}$, when A is acute and

(i) $\cos A = \frac{3}{5}$, (ii) $\cos A = \frac{15}{17}$,
(iii) $= \frac{12}{13}$, (iv) $= \frac{17}{19}$.

Prove that

8. $\sin A = \dfrac{2\tan \dfrac{A}{2}}{1+\tan^2 \dfrac{A}{2}}$.

9. $\sin A = \dfrac{2}{\cot \frac{A}{2} + \tan \frac{A}{2}}$.

10. $\sin A = \dfrac{2 \cot \frac{A}{2}}{1 + \cot^2 \frac{A}{2}}$.

11. $\sin A = \cos^2 \left(\frac{\pi}{4} - \frac{A}{2}\right) - \sin^2 \left(\frac{\pi}{4} - \frac{A}{2}\right)$.

12. $\sin A = \left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)^2 - 1$.

13. $\sin A = 1 - \left(\sin \frac{A}{2} - \cos \frac{A}{2}\right)^2$.

14. $\sin A = \dfrac{\tan \left(\frac{\pi}{4} + \frac{A}{2}\right) - \tan \left(\frac{\pi}{4} - \frac{A}{2}\right)}{\tan \left(\frac{\pi}{4} + \frac{A}{2}\right) + \tan \left(\frac{\pi}{4} - \frac{A}{2}\right)}$.

15. $\cos A = \dfrac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \dfrac{\cot \frac{A}{2} - \tan \frac{A}{2}}{\cot \frac{A}{2} + \tan \frac{A}{2}}$.

16. $\cos A = \dfrac{\cot^2 \frac{A}{2} - 1}{\cot^2 \frac{A}{2} + 1}$.

17. $\cos A = \dfrac{2 - \sec^2 \frac{A}{2}}{\sec^2 \frac{A}{2}}$.

18. $\cos A = 2 \sin \left(\frac{\pi}{4} - \frac{A}{2}\right) \cos \left(\frac{\pi}{4} - \frac{A}{2}\right)$.

19. $\cos 2A = \dfrac{2}{\tan \left(\frac{\pi}{4} + A\right) + \tan \left(\frac{\pi}{4} - A\right)}$.

20. $\tan 2\theta = \dfrac{\sin 4\theta}{1+\cos 4\theta}$.

21. $\tan 2\theta = \dfrac{1-\cos 4\theta}{\sin 4\theta}$.

22. $\tan 2\theta = \dfrac{2}{\cot\theta - \tan\theta}$.

23. $\tan 2A = \dfrac{\cot(45^\circ - A) - \tan(45^\circ - A)}{2}$.

24. $\tan 2\theta = \dfrac{\sin 4\theta - \cos 4\theta + 1}{\sin 4\theta + \cos 4\theta + 1}$.

25. $\tan 2\theta = \dfrac{\sin 4\theta + \sin 2\theta}{1 + \cos 4\theta + \cos 2\theta}$.

26. $\tan\theta = \dfrac{\sin 4\theta}{1+\cos 2\theta}\cdot\dfrac{\cos 2\theta}{1+\cos 4\theta}$.

27. $\tan\theta = \dfrac{\sin 4\theta}{\cos 2\theta}\cdot\dfrac{1-\cos 2\theta}{1-\cos 4\theta}$.

28. $\cos^4 A - \sin^4 A = \cos 2A$.

29. $\cos^4 A + \sin^4 A = \cos^2 2A + \frac{1}{2}\sin^2 2A = 1 - \frac{1}{2}\sin^2 2A$.

30. $\cos^6\alpha + \sin^6\alpha = 1 - \frac{3}{4}\sin^2 2\alpha = \frac{1}{4} + \frac{3}{4}\cos^2 2\alpha$.

31. $\cos^6\alpha - \sin^6\alpha = \cos 2\alpha - \frac{1}{8}\sin 2\alpha \sin 4\alpha$

$$= \cos 2\alpha\,(\cos^2 2\alpha + \tfrac{3}{4}\sin^2 2\alpha).$$

32. $\cos^8 A + \sin^8 A = (\cos^4 A - \sin^4 A)^2 + 2\sin^4 A\cos^4 A$

$$= 1 - \sin^2 2A + \tfrac{1}{8}\sin^4 2A.$$

33. $\cos^8 A - \sin^8 A = \frac{1}{2}\cos 2A\,(1 + \cos^2 2A)$.

34. $4(\cos^6 A + \sin^6 A) - 3(\cos^4 A - \sin^4 A)^2 = 1$.

35. $\cos^6 A - \sin^6 A = \frac{15}{16}\cos 2A + \frac{1}{16}\cos 6A$.

36. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

37. $\cos 4A = 1 - 8\cos^2 A + 8\cos^4 A$.

38. $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A.$

39. $\cos 5A = 5 \cos A - 20 \cos^3 A + 16 \cos^5 A.$

40. $\sin 6A = 2 \sin A \cos A (16 \cos^4 A - 16 \cos^2 A + 3).$

41. $\sin 6A \operatorname{cosec} 2A = 3 - 16 \sin^2 A + 16 \sin^4 A.$

42. $(\cos A + \sin A)^2 = 1 + \sin 2A.$

43. $(\cos A - \sin A)^2 = 1 - \sin 2A.$

44. $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A.$

45. $\dfrac{\cos^3 A + \sin^3 A}{\cos A + \sin A} = 1 - \frac{1}{2} \sin 2A.$

46. $\dfrac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = 1 + \frac{1}{2} \sin 2A.$

47. $1 - \frac{1}{2} \sin 2A \tan A = \cos^2 A.$

48. $\operatorname{cosec} 2A - \tan A = \cot 2A.$

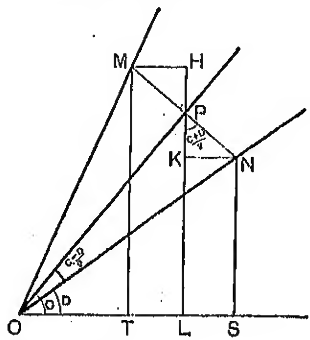
49. $\cot A - \operatorname{cosec} 2A = \cot 2A.$

50. $\dfrac{2 \sin A}{\cos A + \cos 3A} = \tan 2A - \tan A.$

129. The above formulae can be proved geometrically.

$$\left.\begin{array}{r} M\hat{O}S = C \\ N\hat{O}S = D \\ OP \text{ bisects } M\hat{O}N \end{array}\right\} \quad \begin{array}{l} \text{then } P\hat{O}N = \dfrac{C-D}{2} \\ \\ P\hat{O}S = \dfrac{C+D}{2} \end{array}$$

the rest of the figure explains itself.



Let $ON = OM =$ unit of length.

$$\sin C + \sin D = MT + NS = 2PL$$
$$= 2OP \sin \frac{C+D}{2}$$
$$= 2 \cos \frac{C-D}{2} \sin \frac{C+D}{2},$$

$$\sin C - \sin D = MT - NS = 2PK$$
$$= 2PN \cos \frac{C+D}{2} = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2},$$

$$\cos C + \cos D = OT + OS = 2OL$$
$$= 2OP \cos \frac{C+D}{2} = 2 \cos \frac{C-D}{2} \cos \frac{C+D}{2},$$

$$\cos D - \cos C = OS - OT = 2LS = 2KN$$
$$= 2PN \sin \frac{C+D}{2}$$
$$= 2 \sin \frac{C-D}{2} \sin \frac{C+D}{2}.$$

EXAMPLES XXXI.

Express as a product of two trigonometrical ratios:

1. $\sin 2A + \sin A$.
2. $\cos 2A + \cos 3A$.
3. $\cos 5A - \cos 7A$.
4. $\sin 5A - \sin 3A$.
5. $\sin 11A + \sin 5A$.
6. $\cos 3A + \cos 5A$.
7. $\cos A - \cos 5A$.
8. $\sin 3A - \sin 7A$.
9. $\sin 30° + \sin 62°$.
10. $\cos 35° - \cos 55°$.
11. $\cos 42° + \cos 36°$.
12. $\sin 52° - \sin 32°$.
13. $\cos 51° + \cos 23°$.
14. $\sin 15° + \sin 11°$.
15. $\sin 23° - \sin 49°$.
16. $\cos 52° - \cos 42°$.

Prove the following statements:

17. $\dfrac{\cos 2A - \cos 5A}{\sin 2A + \sin 5A} = \tan \dfrac{3A}{2}$.

18. $\dfrac{\cos 2A + \cos A}{\sin 2A - \sin A} = \cot \dfrac{A}{2}$.

19. $\dfrac{\sin 3A + \sin 5A}{\sin 5A - \sin 3A} = \cot A \tan 4A$.

20. $\dfrac{\sin 60° + \sin 30°}{\cos 30° - \cos 60°} = \cot 15°$.

21. $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \dfrac{1}{\sqrt{3}}$.

22. $\dfrac{\cos 20° - \cos 70°}{\sin 70° - \sin 20°} = 1$.

23. $\dfrac{\sin (3A + B) - \sin (A + B)}{\cos (3A + B) + \cos (A + B)} = \tan A$.

24. $\dfrac{\sin (3A + 2B) + \sin A}{\cos A - \cos (3A + 2B)} = \cot (A + B)$.

25. $\dfrac{\cos 3B - \cos(4A + 3B)}{\sin(4A + 3B) + \sin 3B} = \tan 2A.$

26. $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A.$

27. $\dfrac{\sin 4A - \sin 2A}{\cos 4A + \cos 2A} = \tan A.$

28. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$

EXAMPLES XXXII.

Express as the sum or difference of two trigonometrical ratios:

1. $2 \sin 2A \cos A.$
2. $2 \cos 2A \cos A.$
3. $2 \sin A \cos 4A.$
4. $2 \sin A \sin 3A.$
5. $2 \sin 4A \cos 8A.$
6. $2 \cos 5A \cos 7A.$
7. $2 \cos 5A \sin 3A.$
8. $2 \sin 3A \sin 5A.$

Express as the sum or difference of two trigonometrical ratios and then find the values from tables:

9. $2 \cos 60° \sin 30°.$
10. $2 \cos 45° \cos 53°.$
11. $\sin 35° \cos 45°.$
12. $2 \cos 50° \cos 70°.$
13. $\sin 52° \sin 75°.$
14. $2 \sin 55° \cos 40°.$
15. $\cos 32° \cos 58°.$
16. $\cos 140° \sin 73°.$

Express as the sum or difference of two trigonometrical ratios:

17. $2 \cos(2A + B) \cos(A - B).$
18. $2 \sin(A + 3B) \sin(2A + 5B).$
19. $2 \cos(x + 2y) \sin(3x + 4y).$
20. $2 \cos(3x + 5y) \sin(x - y).$

Some useful illustrative examples, which are of frequent occurrence, follow.

130. If $A+B+C=180^\circ$, *i.e.* if A, B, and C are the angles of a triangle,

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$$

[*Note to the student.*

Remember
$$\sin(180^\circ-A)=\sin A,$$
$$\therefore\ \sin(B+C)=\sin A.$$
$$\cos(180^\circ-A)=-\cos A,$$
$$\therefore\ \cos(B+C)=-\cos A.]$$

1st Method.

$$\sin 2B+\sin 2C=2\sin(B+C)\cos(B-C)$$
$$=2\sin A\cos(B-C)$$
$$\sin 2A=2\sin A\cos A$$
$$=-2\sin A\cos(B+C).$$
$$\therefore\ \sin 2A+\sin 2B+\sin 2C=2\sin A\{\cos(B-C)-\cos(B+C)\}$$
$$=2\sin A\{2\sin B\sin C\}$$
$$=4\sin A\sin B\sin C.$$

2nd Method.

$$4\sin A\sin B\sin C=2\sin A\{\cos(B-C)-\cos(B+C)\}$$
$$=2\sin A\cos(B-C)-2\sin A\cos(B+C)$$
$$=2\sin(B+C)\cos(B-C)+2\sin A\cos A$$
$$=\sin 2B+\sin 2C+\sin 2A.$$

131. If $A+B+C=180^\circ$,

$$\cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C+1=0.$$

1st Method.

$$\cos 2A+\cos 2B+\cos 2C=2\cos^2 A-1+2\cos(B+C)\cos(B-C)$$
$$=2\cos^2 A-1-2\cos A\cos(B-C)$$
$$=2\cos A\{\cos A-\cos(B-C)\}-1$$
$$=-2\cos A\{\cos(B+C)+\cos(B-C)\}-1$$
$$=-4\cos A\cos B\cos C-1.$$
$$\therefore\ \cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C+1=0.$$

2nd Method.

$$4\cos A\cos B\cos C = 2\cos A\,(2\cos B\cos C)$$
$$= 2\cos A\,\{\cos(B+C)+\cos(B-C)\}$$
$$= -2\cos^2 A + 2\cos A\cos(B-C)$$
$$= -(1+\cos 2A) - 2\cos(B+C)\cos(B-C)$$
$$= -1-\cos 2A-\cos 2B-\cos 2C.$$

$$\therefore\ \cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C+1=0.$$

132. If $A+B+C=180°$,

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

[*Note to the student.*

Remember $\qquad \sin(90°-\theta)=\cos\theta,$

$$\therefore\ \sin\frac{B+C}{2}=\cos\frac{A}{2},$$

and
$$\cos\frac{B+C}{2}=\sin\frac{A}{2}.]$$

1st Method.

$$\sin B+\sin C=2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$$
$$=2\cos\frac{A}{2}\cos\frac{B-C}{2},$$

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$$
$$=2\cos\frac{B+C}{2}\cos\frac{A}{2}.$$

$$\therefore\ \sin A+\sin B+\sin C=2\cos\frac{A}{2}\left\{\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right\}$$
$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

2nd Method.

$$4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=2\cos\frac{A}{2}\left\{2\cos\frac{B}{2}\cos\frac{C}{2}\right\}$$

$$=2\cos\frac{A}{2}\left\{\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right\}$$

$$=2\cos\frac{A}{2}\sin\frac{A}{2}+\sin\frac{B+C}{2}\cos\frac{B-C}{2}$$

$$=\sin A+\sin B+\sin C.$$

133. If $A+B+C=180°$,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

1st Method.

$$\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

But $$\tan(A+B+C)=\tan 180°=0.$$

$$\therefore\ \tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

2nd Method.

$$\tan(180°-\theta)=-\tan\theta.$$

$$\therefore\ \tan(B+C)=-\tan A.$$

$$\therefore\ \frac{\tan B+\tan C}{1-\tan B\tan C}=-\tan A.$$

$$\therefore\ \tan B+\tan C=-\tan A+\tan A\tan B\tan C.$$

$$\therefore\ \tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

134. If $\alpha+\beta+\gamma=0$,

$$\sin\alpha+\sin\beta+\sin\gamma=-4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

[*Note.* Remember,

$$\sin(-\theta)=-\sin\theta,$$

$$\cos(-\theta)=\cos\theta,$$

and since $$\frac{\beta+\gamma}{2}=-\frac{\alpha}{2},$$

$$\therefore\ \sin\frac{\beta+\gamma}{2}=-\sin\frac{\alpha}{2},$$

$$\cos\frac{\beta+\gamma}{2}=\cos\frac{\alpha}{2}.]$$

1st Method.

$$\sin\beta + \sin\gamma = 2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}$$
$$= -2\sin\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}.$$

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$
$$= 2\sin\frac{\alpha}{2}\cos\frac{\beta+\gamma}{2}.$$

$$\therefore\ \sin\alpha + \sin\beta + \sin\gamma = 2\sin\frac{\alpha}{2}\left\{\cos\frac{\beta+\gamma}{2} - \cos\frac{\beta-\gamma}{2}\right\}$$
$$= -4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

2nd Method.

$$4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 2\sin\frac{\alpha}{2}\left\{\cos\frac{\beta-\gamma}{2} - \cos\frac{\beta+\gamma}{2}\right\}$$
$$= -2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2} - 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$
$$= -\sin\beta - \sin\gamma - \sin\alpha.$$

N.B. The above might be written

$$\sin(y-z) + \sin(z-x) + \sin(x-y) = -4\sin\frac{y-z}{2}\sin\frac{z-x}{2}\sin\frac{x-y}{2},$$

x, y and z being any angles.

135. The above five examples are for angles with certain relations, the two following are perfectly general.

To prove $\sin\theta + \sin\phi + \sin\psi - \sin(\theta+\phi+\psi)$

$$= 4\sin\frac{\theta+\phi}{2}\sin\frac{\phi+\psi}{2}\sin\frac{\psi+\theta}{2}.$$

$$4\sin\frac{\theta+\phi}{2}\sin\frac{\phi+\psi}{2}\sin\frac{\psi+\theta}{2}$$
$$= 2\sin\frac{\theta+\phi}{2}\left\{\cos\frac{\theta-\phi}{2} - \cos\frac{\theta+\phi+2\psi}{2}\right\}$$
$$= 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} - 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi+2\psi}{2}$$
$$= \sin\theta + \sin\phi - \{\sin(\theta+\phi+\psi) - \sin\psi\}$$
$$= \sin\theta + \sin\phi + \sin\psi - \sin(\theta+\phi+\psi).$$

136. To prove $\cos\theta + \cos\phi + \cos\psi + \cos(\theta+\phi+\psi)$

$$= 4\cos\frac{\phi+\psi}{2}\cos\frac{\psi+\theta}{2}\cos\frac{\theta+\phi}{2}.$$

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$

$$\cos(\theta+\phi+\psi) + \cos\psi = 2\cos\frac{\theta+\phi+2\psi}{2}\cos\frac{\theta+\phi}{2}.$$

$\therefore\ \cos\theta + \cos\phi + \cos\psi + \cos(\theta+\phi+\psi)$

$$= 2\cos\frac{\theta+\phi}{2}\left\{\cos\frac{\theta+\phi+2\psi}{2} + \cos\frac{\theta-\phi}{2}\right\}$$

$$= 2\cos\frac{\theta+\phi}{2}\cdot 2\cos\frac{\phi+\psi}{2}\cos\frac{\psi+\theta}{2}$$

$$= 4\cos\frac{\phi+\psi}{2}\cos\frac{\psi+\theta}{2}\cos\frac{\theta+\phi}{2}.$$

EXAMPLES XXXIII.

Prove the following when $A + B + C = 180°$:

1. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

2. $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$.

3. $\cos A + \cos B - \cos C = 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} - 1$.

4. $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$.

5. $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C$.

Imp 6. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

7. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$

8. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}.$

9. $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$

10. $\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C).$

11. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C.$

12. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C.$

13. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C.$

14. $\sin^2 2A+\sin^2 2B+\sin^2 2C=2(1-\cos 2A\cos 2B\cos 2C).$

15. $\cos^2 2A+\cos^2 2B+\cos^2 2C=1+2\cos 2A\cos 2B\cos 2C.$

16. $\sin 4A+\sin 4B+\sin 4C=-4\sin 2A\sin 2B\sin 2C.$

17. $\cos 4A+\cos 4B+\cos 4C=4\cos 2A\cos 2B\cos 2C-1.$

18. $\sin 4A+\sin 4B-\sin 4C=-4\cos 2A\cos 2B\sin 2C.$

19. $\cos 4A+\cos 4B-\cos 4C=4\sin 2A\sin 2B\cos 2C+1.$

20. $\dfrac{\sin A+\sin B-\sin C}{\sin A-\sin B+\sin C}=\tan\frac{B}{2}\cot\frac{C}{2}.$

21. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin 2A+\sin 2B-\sin 2C}=\tan A\tan B.$

22. $\dfrac{\cos A+\cos B+\cos C-1}{\cos A+\cos B-\cos C+1}=\tan\frac{A}{2}\tan\frac{B}{2}.$

23. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

24. $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C.$

25. $\dfrac{\tan A}{\tan B}+\dfrac{\tan B}{\tan A}+\dfrac{\tan B}{\tan C}+\dfrac{\tan C}{\tan B}+\dfrac{\tan C}{\tan A}+\dfrac{\tan A}{\tan C}$

$=\cos A\sec B\sec C+\sec A\cos B\sec C+\sec A\sec B\cos C.$

Prove the following identities:

26. $\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)$
$+\cos(-\alpha+\beta+\gamma)=4\cos\alpha\cos\beta\cos\gamma.$

27. $\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)-\cos(-\alpha+\beta+\gamma)$
$-\cos(\alpha+\beta+\gamma)=4\cos\alpha\sin\beta\sin\gamma.$

28. $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)$
$=\sin(\alpha+\beta+\gamma)+4\sin\alpha\sin\beta\sin\gamma.$

29. $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta+\gamma)$
$=\sin(\alpha+\beta-\gamma)+4\cos\alpha\cos\beta\sin\gamma.$

30. $\tan(y-z)+\tan(z-x)+\tan(x-y)$
$=\tan(y-z)\tan(z-x)\tan(x-y).$

31. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2(\alpha+\beta+\gamma)$
$=2[1+\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)].$

32. $\tan(A+60°)\tan(A-60°)+\tan A\tan(A+60°)$
$+\tan A\tan(A-60°)=-3.$

33. $\cos A+\cos B+\cos C+\cos(A+B+C)$
$=4\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(C+A)\cos\frac{1}{2}(A+B).$

34. $\cos^2 A+\cos^2 B+\cos^2(A+B)-2\cos A\cos B\cos(A+B)=1.$

35. $\tan A+2\tan 2A+4\cot 4A=\cot A.$

36. $\sin^2 A+\sin^2 B+\sin^2 C+2\sin A\sin B\sin C=1,$
if $A+B+C=90°.$

37. $\tan A+\cos A\sec B\sec C=\tan B+\cos B\sec A\sec C$
$=\tan C+\cos C\sec A\sec B,$
if $A+B+C=90°.$

38. $\cos 2A+\cos 2B+\cos 2C+\cos 2D$
$=4[\cos A\cos B\cos C\cos D-\sin A\sin B\sin C\sin D],$
if $A+B+C+D=0.$

39. $\Sigma\sin^3 A\cos(B-C)=3\sin A\sin B\sin C,$
if $A+B+C=180°.$

40. $\sin^2(A-B)\cos A+\cos(2A-B)\sin(C-A)\sin(C-B)$
$=\sin^2(A-C)\cos A+\cos C\sin(C-B)\sin(A-B).$

Miscellaneous Examples on Chapters XI and XII start in Test Paper XI, page 231.

CHAPTER XIII.

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE (*continued*).

In Chapter VIII the following formulae connecting the sides and angles of a triangle are proved.

1. $$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

2. $$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

3. $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

4. Area of a triangle $=\sqrt{s(s-a)(s-b)(s-c)}$.

5. $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

They are now proved by the aid of Chapters XI, XII.

137. By Art. 75

$$\cos A=\frac{b^2+c^2-a^2}{2bc},$$

$$\therefore\ 1-2\sin^2\frac{A}{2}=\frac{b^2+c^2-a^2}{2bc},$$

$$\therefore\ \sin^2\frac{A}{2}=\frac{2bc-b^2-c^2+a^2}{4bc}$$

$$=\frac{a^2-(b-c)^2}{4bc}=\frac{(a-b+c)(a+b-c)}{4bc}$$

$$=\frac{(s-b)(s-c)}{bc}\text{ where }2s=a+b+c,$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

the positive sign being taken because A is $< 180°$,

$$\therefore \sin\frac{A}{2} \text{ is positive.}$$

Also $\quad 2\cos^2\frac{A}{2} - 1 = \frac{b^2+c^2-a^2}{2bc},$

$$\therefore \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$= \frac{s(s-a)}{bc},$$

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

the positive sign being taken because A is $< 180°$,

$$\therefore \cos\frac{A}{2} \text{ is positive.}$$

Hence $\quad \tan\frac{A}{2} = \dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}.$

Notice $\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)},$$

the same result as that obtained in Art. 77.

138. The area of a triangle $\Delta = \frac{1}{2}bc\sin A$ (Art. 78)

$$= bc\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= bc\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

139. By Art. 72

$$\frac{\sin B}{\sin C}=\frac{b}{c},$$

$$\therefore \quad \frac{\sin B-\sin C}{\sin B+\sin C}=\frac{b-c}{b+c},$$

$$\therefore \quad \frac{2\sin\dfrac{B-C}{2}\cos\dfrac{B+C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}}=\frac{b-c}{b+c},$$

$$\therefore \quad \tan\frac{B-C}{2}=\frac{b-c}{b+c}\tan\frac{B+C}{2}$$

$$=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

140. Many formulae are not suitable for logarithmic calculation; such formulae can often be transformed into two others suitable for such calculation by the aid of a "*subsidiary*" angle.

141. To transform

$$a^2=b^2+c^2-2bc\cos A.$$

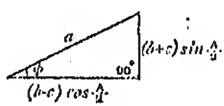
(i) $$a^2=(b^2+c^2)\left(\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right)-2bc\left(\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right)$$

$$=(b-c)^2\cos^2\frac{A}{2}+(b+c)^2\sin^2\frac{A}{2},$$

therefore we may draw a right-angled triangle as in the fig.

Hence

$$\tan\phi=\frac{(b+c)\sin\dfrac{A}{2}}{(b-c)\cos\dfrac{A}{2}}=\frac{b+c}{b-c}\tan\frac{A}{2},$$



from which ϕ may be *logarithmically* calculated, and then

$$a=(b-c)\cos\frac{A}{2}\sec\phi$$

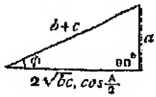
from which a may be logarithmically calculated.

(ii) $$a^2 = b^2 + c^2 - 2bc\left(2\cos^2\frac{A}{2} - 1\right)$$

$$= (b+c)^2 - 4bc\cos^2\frac{A}{2},$$

therefore we may draw a right-angled triangle as in the fig.

Hence
$$\cos\phi = \frac{2\sqrt{bc}\,.\cos\frac{A}{2}}{b+c},$$



from which ϕ may be logarithmically calculated, and then

$$a = (b+c)\sin\phi,$$

from which a may be logarithmically calculated.

(iii) $$a^2 = b^2 + c^2 - 2bc\left(1 - 2\sin^2\frac{A}{2}\right)$$

$$= (b-c)^2 + 4bc\sin^2\frac{A}{2},$$

therefore we may draw a right-angled triangle as in the fig.

Hence

$$\tan\phi = \frac{2\sqrt{bc}\,.\sin\frac{A}{2}}{b-c},$$

from which ϕ may be logarithmically calculated, and then

$$a = (b-c)\sec\phi,$$

from which a may be logarithmically calculated.

Ex. 1. Given $b=71$; $c=35$; $A=29° 34'$; find a.

From (i), $$L\tan\phi=\log(b+c)-\log(b-c)+L\tan\frac{A}{2},$$

$$\begin{aligned} \log 106 &= 2\cdot0253 \\ L\tan 14° 47' &= 9\cdot4215 \\ &\overline{11\cdot4468} \\ \log 36 &= 1\cdot5563 \\ \therefore\ L\tan\phi &= 9\cdot8905 \\ \therefore\ \phi &= 37° 51' \end{aligned}$$

and $$\log a=\log(b-c)+L\cos\frac{A}{2}+L\sec\phi-20$$

$$\begin{aligned} \log 36 &= 1\cdot5563 \\ L\cos 14° 47' &= 9\cdot9853 \\ L\sec 37° 51' &= 10\cdot1026 \\ &\overline{21\cdot6442} \\ \therefore\ \log a &= 1\cdot6442 \\ \therefore\ a &= 44\cdot08. \end{aligned}$$

Ex. 2. Prove that $\dfrac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\dfrac{a-b}{c}$.

$$\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\frac{\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}-\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}}{\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}}$$

$$=\frac{\dfrac{\Delta}{s(s-a)}-\dfrac{\Delta}{s(s-b)}}{\dfrac{\Delta}{s(s-a)}+\dfrac{\Delta}{s(s-b)}}$$

$$=\frac{\dfrac{1}{s-a}-\dfrac{1}{s-b}}{\dfrac{1}{s-a}+\dfrac{1}{s-b}}=\frac{s-b-s+a}{s-b+s-a}$$

$$=\frac{a-b}{c}.$$

EXAMPLES XXXIV.

Prove that

1. $s\tan\frac{A}{2}=(s-b)\cot\frac{C}{2}=(s-c)\cot\frac{B}{2}$.

2. $(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$.

3. $\Delta^2=abcs\,.\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

4. $\dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=\dfrac{\Delta}{s}$.

5. $\frac{1}{a}\cos^2\frac{A}{2}+\frac{1}{b}\cos^2\frac{B}{2}+\frac{1}{c}\cos^2\frac{C}{2}=\frac{s^2}{abc}$.

6. $(b+c-a)\sin\frac{A}{2}=2a\sin\frac{B}{2}\sin\frac{C}{2}$.

7. $(a+b+c)\sin\frac{A}{2}=2a\cos\frac{B}{2}\cos\frac{C}{2}$.

8. $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=s$.

9. $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0$.

10. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

11. $\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1$.

12. In any triangle prove $c=(a-b)\cos\frac{C}{2}\sec\phi$

where $$\tan\phi=\frac{a+b}{a-b}\tan\frac{C}{2}.$$

Hence find ϕ and c, given

$$a=17;\quad b=13;\quad C=47°\,14'.$$

13. In any triangle prove $c=(a+b)\cos\phi$

where $$\sin\phi=\frac{2\sqrt{ab}}{a+b}\cos\frac{C}{2}.$$

Hence find ϕ and c, given

$$a=11;\quad b=25;\quad C=106°\,16'.$$

14. In any triangle prove $c=(a-b)\sec\phi$

where $$\tan\phi=\frac{2\sqrt{ab}}{a-b}\sin\frac{C}{2}.$$

Hence find ϕ and c, given

$$a=54;\quad b=34;\quad C=45°\,12'.$$

TEST PAPERS.

[*Including Measurement of Angles, πr^2, $2\pi r$, $r\theta$, $\frac{1}{2}r^2\theta$, Constructions of Angles with given ratios, etc. Chapters I and II.*]

$$\pi = \frac{22}{7}.$$

I.

1. Express 17° 15′ 42″ as the decimal of a right angle.

2. Find the number of radians (correct to 4 places of decimals) in 15°, and express 2·4 radians in degrees.

3. Find to the nearest square centimetre the area of a circle with a radius of 5 metres.

4. An arc of a circle is 6 metres; find the number of radians it subtends at the centre if the radius is 5 metres.

5. The angle subtended at the centre of a circle of radius 4 centimetres by a certain arc is 22°. Find the length of the arc.

6. Draw with a protractor an angle of 52°; find by measurement the values of sin 52° and cos 52°, thence deduce roughly that

$$\sin^2 52° + \cos^2 52° = 1.$$

7. Given that $\sin \alpha = \cdot 72$, construct the angle and then measure it with a protractor to the nearest degree.

8. If $\sin A = \frac{1}{3}$, find cos A and tan A (correct to 2 places of decimals).

II.

1. Express ·7245 of a right angle in degrees, minutes and seconds.

2. Express $\frac{5\pi}{6}$ radians in degrees and 32° in radians (correct to 4 places of decimals).

3. Find (i) the circumference, (ii) the area of a circle of radius 7 centimetres.

4. Find the number of degrees subtended at the centre of a circle of radius 3 metres by an arc of 5 metres.

5. Find the area of a sector of a circle of radius 4 centimetres containing an angle of 1·5 radians.

6. If the cosine of a certain angle is ·35, construct the angle and then measure it to the nearest degree.

7. Draw an angle of 29° and find by measurement sin 29°, cos 29°, tan 29°; thence deduce roughly that $\tan 29° = \frac{\sin 29°}{\cos 29°}$.

8. If $\cos A = \frac{2}{5}$, find cosec A and cot A. (Answer correct to 3 places of decimals.)

III.

1. What decimal of 1 right angle is 52° 16′ 50″?

2. Express 1·6 radians in degrees and 75° in radians (correct to 4 places).

3. If the circumference of a circle is 20 metres, find the radius in metres.

4. The area of a circle is 50 square metres; find the radius to the nearest hundredth of a metre.

5. The area of a sector of a circle of 5 metres radius is 7 square metres; find the size of the angle of the sector in radians.

6. Draw an angle of 73° and find by measurement the values of sin 73° and cosec 73°. Thus show roughly that

$$\sin 73° \times \operatorname{cosec} 73° = 1.$$

7. Given that $\tan x = 2·5$, construct and then measure x to the nearest degree.

8. If $\sin x = \frac{3}{5}$, find $\cos x$, $\cot x$ and $\sec x$.

IV.

1. Express 1·245 of a right angle in degrees and minutes.

2. If $\cot A = \frac{9}{2}$, find $\cos A$ and $\operatorname{cosec} A$ (answer correct to 3 places of decimals).

3. With your instruments make an angle whose tangent is 0·73 and then measure it to the nearest degree.

4. My compasses have legs 10 cms. long. I open them out to an angle of 35° and describe a circle. Find from a carefully drawn diagram the distance between the points of the compasses (to the nearest mm.) and calculate the area of the circle (to the nearest sq. cm.).

5. If the area of a circle is 60 sq. cms., find the radius to the nearest millimetre.

6. Two angles of a triangle are 1·3 radians and 62° 30′. Find the third angle in degrees.

7. Find in degrees the angle whose radian measure is $\frac{5}{8}$.

8. In the triangle ABC, $\tan B = \frac{4}{3}$, $\tan C = \frac{8}{15}$. Find the ratio of AB to AC.

V.

1. Given that $\sin A = \frac{12}{37}$; find the other trigonometrical ratios of A.

2. If the radius of a circle be 25 metres, find to 3 decimal places the length of the arc subtending an angle of 3° at the centre. ($\pi = 3·1416$.)

3. If in a triangle ABC, $CA = CB = 2$ and $AB = 3$; find the value of $(\sin A - \cos A)(\sec A + \operatorname{cosec} A)$, correct to 2 places of decimals.

4. With your instruments construct an angle whose sine is 0·6. Bisect the angle and from the figure measure off the value of $\sin \frac{A}{2}$. (Answer correct to 2 places of decimals.)

5. Draw an angle of 35° and find by measurement the values of sin 35° and cos 35°. Thence show roughly that

$$\sin^2 35° + \cos^2 35° = 1.$$

6. Find the number of radians in 27° 15′. (Answer correct to 3 places of decimals.)

7. Find to the nearest sq. millimetre the area of a circle with a radius of 4 centimetres.

8. If an angle contains A seconds and its circular measure is a, show that approximately $A = 206265 \times a$.

VI.

1. Express ·8245 of a right angle in degrees, minutes and seconds.

2. With your instruments make an angle whose secant is 7·2 and then measure it to the nearest degree.

3. What is the measure (i) in degrees, (ii) in radians (correct to 2 decimal places) of an internal angle of a regular heptagon?

4. Find the number of radians (correct to 2 places of decimals) in one of the angles of a regular figure of 33 sides.

5. Given that $\tan A = \frac{3}{7}$; find the other trigonometrical ratios of A (correct to 2 places of decimals).

6. Draw an angle of 27° and find by measurement the values of tan 27° and sec 27°. Thence show roughly that

$$1 + \tan^2 27° = \sec^2 27°.$$

7. Two angles of a triangle are 23° and 77° 10′. Find (correct to 2 places of decimals) the number of radians in the third angle.

8. Find the number of degrees, minutes and seconds in an angle which contains 1·724 radians. (Answer to the nearest second.)

VII.

1. If there are 11 spokes in a cart-wheel, express the angle between them in (i) degrees, (ii) radians (correct to 2 places of decimals).

2. Draw an angle of 40° and find by measurement the values of cot 40° and cosec 40°. Thence show roughly that

$$1 + \cot^2 40° = \operatorname{cosec}^2 40°.$$

3. Construct an angle whose cotangent is 1·8 and then measure the angle to the nearest degree.

4. Two angles of a triangle are 1·2 and 0·8 radians respectively. Calculate the remaining angle to the nearest degree.

5. Draw angles of 32° and 58°; find by measurement their sines and cosines, and thus deduce roughly that

$$\sin 32° = \cos 58°.$$

6. If A, B and C are the three angles of a triangle and $A = 2B = 3C$, express each of them in degrees.

7. Find the number of degrees and radians (correct to 2 places of decimals) between the positions of the large hand of a clock at 1.10 and 1.20.

8. If the area of a circle is 40 sq. centimetres, find (to the nearest millimetre) the length of an arc subtending an angle of 1·5 radians at the centre.

VIII.

1. Find the number of degrees and radians (correct to 2 places of decimals) between the positions of the large hand of a watch at 2.20 and 2.42.

2. Construct an angle of 51° and find by measurement the values of sin 51° and cos 51°. Thence show roughly that

$$\cos^2 51° = 1 - \sin^2 51°.$$

3. Construct an angle whose cosecant is 5·2, and then measure it to the nearest degree.

4. The length of an arc subtending an angle of 10° at the centre of a circle is 10 centimetres. Find the area of the circle to the nearest square millimetre.

5. Construct a triangle the sides of which are 6·5, 5·2, and 3·9 centimetres respectively. Thence determine the sines of the angles opposite the second and third sides and show that they are proportional to the second and third sides.

6. Draw angles of 24° and 48°, and thence prove roughly by measurement that

$$\sin 48^\circ = 2 \sin 24^\circ \cos 24^\circ.$$

7. Find in degrees and radians (correct to 2 places of decimals) the angle of a regular polygon of 21 sides.

8. A circular grass plot has a radius of 24 metres, and round this is a gravel path of width 1·2 metres. What is the area of the path to the nearest square metre?

[*Including Simple Identities, Angles of 0°, 30°, 45° etc., Use of Tables, Complementary Angles and Easy Equations. Chapters III and IV.*]

IX.

1. The difference of two angles is 10°, and the radian measure of their sum is 2; find the number of radians in each angle.

2. Draw angles of 33° and 57° and find by measurement sin 33°, cos 33°, sin 57° and cos 57°. Thence prove roughly that

$$\sin 33^\circ \cos 57^\circ + \cos 33^\circ \sin 57^\circ = 1.$$

3. If $\cos A = a$, prove that

$$\tan A = \frac{\sqrt{1 - a^2}}{a}.$$

4. With instruments, construct an angle A whose cosine is 0·6. Bisect the angle A, and from your figure measure off the value of $\cos \frac{A}{2}$.

Compare the value so obtained with that given by the Tables.

5. The range of a certain gun is 1000 sin 2A metres, where A is the elevation of the gun. Find from the tables the values of 1000 sin 2A when A has the values 10°, 15°, 20°, 25°, 30°, 35°, 40°, 45°, and draw a curve showing how the range varies as A increases from 10° to 45°.

6. Find the value (correct to 2 places of decimals) of

$$\sin^2 60^\circ + \cos^2 45^\circ + \tan 30^\circ.$$

Delhi [illegible] 1946

7. Prove that

$$\sin^3 A \cos A + \cos^3 A \sin A = \sin A \cos A.$$

Delhi [illegible] 1948

8. Solve the equation

$$\sin \theta + 2 \cos \theta = 1.$$

X.

1. Find the distance in miles between two places on the Equator which differ in longitude by 15° 30′, the earth's equatorial diameter being 7920 miles.

2. If the circumference of a bicycle wheel is 9 feet; through how many degrees does a particular spoke turn as the bicycle goes 150 feet?

3. Find to the nearest minute the angle whose circular measure is $\frac{3}{4}$. Find its tangent from the Tables. Then construct the angle to the nearest degree and find by actual measurement the value of its sine.

4. Prove that

$$(\tan A + \cot A) \sin A \cos A = 1.$$

5. The cosine of an acute angle is $\frac{4}{7}$; find the angle to the nearest minute from the tables; find its sine also from the tables, and verify the result by means of the formula

$$\sin^2 A = 1 - \cos^2 A.$$

6. Find the value of

$$\tan^2 60^\circ + \cot^2 45^\circ + \sin 30^\circ.$$

7. Solve the equation

$$4 \cos^2 \theta - 4 \sin \theta = 1.$$

8. The strength of an electric current determined from a sine galvanometer is $11 \cdot 2 \times \sin A$. Find from the tables the strength of the current when A has the values 10°, 15°, 20°, 25°, 30° and draw a curve showing how the strength varies as A increases from 10° to 30°.

XI.

1. Find, by means of the tables, the value of

$$\sin A \cos B + \cos A \sin B,$$

when $A = 50^\circ$ and $B = 10^\circ$.

Compare your result with the value of $\sin 60^\circ$.

2. Find the number of degrees, minutes and seconds in an angle containing 2·717 radians.

3. Make a triangle ABC having $\hat{C} = 90^\circ$, $CA = 6·1$ cm., $CB = 9·8$ cm. From BC cut off $BQ = 8·6$ cm., and draw QP perpendicular to BC to meet BA in P. From BA cut off BS 9·5 cm., and draw SR perpendicular to BA to meet BC in R. Measure AB, PQ, PB, RS, RB, and so find the ratios $\frac{AC}{AB}$, $\frac{PQ}{PB}$, $\frac{RS}{RB}$ correct to two decimal places.

From these ratios and your tables find the angle B.

4. Tabulate the values of $\sin A - \cos A$ (correct to two places of decimals) when $A = 0^\circ, 20^\circ, 40^\circ, 60^\circ, 80^\circ$.

5. Prove that $2 \cos^2 \theta - 1 = \dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$.

6. Solve the equation

$$4 \cos^2 x + 12 \cos x = 7.$$

7. Prove that

$$\sin A \tan A \sin (90^\circ - A) + \cos A \cot A \cos (90^\circ - A) = 1.$$

8. Determine with your instruments whether

$$\sin 2A \gtreqless 2 \sin A,$$

taking A as 35°. Draw a large figure and indicate the nature of your test.

XII.

1. Prove that

$$\frac{\operatorname{cosec} A - \cot A}{\operatorname{cosec} A + \cot A} + \frac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A} = 2 + 4 \cot^2 A.$$

2. Solve the equation

$$\cos^2 \theta + \cos \theta = \sin^2 \theta.$$

3. Find (correct to 4 significant figures) the number of radians in $1^\circ 30'$ and then finding $\tan 1^\circ 30'$ from the tables, calculate the value of

$$\frac{\tan 1^\circ 30'}{\text{no. of radians in } 1^\circ 30'}$$

correct to 4 places of decimals.

4. If $\sin \theta = \frac{8}{17}$, prove that

$$\tan \theta + \sec \theta = \tfrac{5}{3}.$$

5. If $\sec \theta - \tan \theta = a$, show that

$$\sin \theta = 1 \text{ or } \frac{1 - a^2}{1 + a^2}.$$

6. Supposing the earth to be a sphere of radius 3980 miles, find the length of a meridian arc which subtends an angle of 1″ at the centre. Answer to $\frac{1}{10}$ of a mile.

7. If $A = 60^\circ$, $B = 30^\circ$, $C = 90^\circ$, find the value of

$$\sin A \sin B \sin C + \sin A \cos B \cos C$$
$$- \sin B \cos C \cos A + \sin C \cos A \cos B.$$

8. Find from the tables the value of $\sin \frac{A}{2}$ which makes

$$\sin A = \tfrac{1}{7}.$$

XIII.

1. Prove that

$$\frac{\sin A + \cos A}{\sin A - \cos A} = \frac{1 + 2 \sin A \cos A}{1 - 2 \cos^2 A}.$$

2. Solve the equation

$$\cos^2 x + 5 \sin x \cos x = 3.$$

3. At a point whose distance from the sun's centre is one million of miles, the sun would subtend an angle of $51^\circ 12'$. Calculate approximately the diameter of the sun correct to 1000 miles.

4. Find, by means of Tables, the value (correct to 3 places of decimals) of

$$13{\cdot}5 \times \tan 12° 25'.$$

5. If the cosecant of an angle is 3·42, construct it and measure to the nearest degree. From the figure determine the value of the cotangent of the angle; thence show, approximately, that

$$\operatorname{cosec}^2 x = 1 + \cot^2 x.$$

6. Tabulate (correct to 2 places of decimals) the values of $\sin A + 2\cos A$ when $A = 0°, 10°, 20°, 30°, 40°, 50°$.

7. If the circumference of a circular grass plot is 300 metres, find the area, correct to the nearest square metre.

8. If an angle A contains 35°, find by measurement the values of $\sin A$, $\sin \frac{A}{2}$, $\cos \frac{A}{2}$. Thence show roughly that

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}.$$

XIV.

1. Prove that

$$\frac{1 - \sin A}{1 - \sec A} - \frac{1 + \sin A}{1 + \sec A} = 2 \cot A (\cos A - \operatorname{cosec} A).$$

2. The tangent of an angle is 2·4. Find by measurement the values of the cosecant of the angle and the cosecant of the complement of half the angle.

3. Having given that $\sin A = \frac{4}{5}$ and $\sin B = \frac{8}{17}$ find the value of

$$(\tan A + \tan B)/(1 - \tan A \tan B).$$

4. Find correct to 3 places of decimals, the number of degrees in an angle containing ·63 radians.

5. A wire AB, 12 metres in length, is bent so as to form an arc of a circle whose diameter is 4 metres; find the angle subtended at the centre of the circle by the chord AB.

Solve the equations

$$\text{(i)} \quad \tan^4\theta - 4\tan^2\theta + 3 = 0,$$
$$\text{(ii)} \quad 3\sin\theta - 3\sin^2\theta = \cos^2\theta.$$

Find to the nearest decimetre the radius of a circle whose s 1 sq. kilometre.

If $\quad \cos\theta + \cos\phi = a \quad$ and $\quad \sin\theta + \sin\phi = b,$

that $\quad \cos\theta\cos\phi + \sin\theta\sin\phi = \dfrac{a^2 + b^2 - 2}{2}.$

XV.

Prove that

$$\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \frac{2\sin^2\theta - 1}{2\sin\theta\cos\theta + 1}.$$

Solve the equations

$$\text{(i)} \quad \cot^4\theta - 4\cot^2\theta + 3 = 0,$$
$$\text{(ii)} \quad 3\tan^2\theta - 7\sec\theta + 5 = 0.$$

If $\tan A = \dfrac{2\sqrt{p}}{1-p}$ find the value of $\cos A$.

If $\tan\theta + \sec\theta = 2$, find $\sin\theta$.

Find the number of radians in the angle between the of a watch at m minutes past 12 o'clock.

Find the value of

$$\cot^2 60° + \sin^2 45° - \cos^2 30°.$$

Find by means of Tables the value of

$$\cos A \cos B + \sin A \sin B,$$
$$A = 60° \quad \text{and} \quad B = 20°.$$

ompare the result with the value of $\cos 40°$.

On squared paper draw a circle of 2 centimetres radius stimate its area, approximately, by counting the number of squares enclosed within its circumference.

ivon that the area of a circle $= \pi$ times the square of its s, calculate from your result the value of π.

y how much per cent. is your result in error, the true value being 3·14159 to 5 decimal places?

XVI.

1. Prove that

(i) $(\cot\theta + \cos\theta)^2 - (\cot\theta - \cos\theta)^2 = 4\sqrt{\cot^2\theta - \cos^2\theta}$,

(ii) $(\tan^2 A - \cot^2 A)/(\sin^2 A - \cos^2 A) = \sec^2 A \operatorname{cosec}^2 A$.

2. Solve the equations

(i) $9\cot^4 x = 1$,

(ii) $\tan^2 x = 2(1 + 2\cos^2 x)$.

3. Find the number of radians in one of the angles of a regular figure of 44 sides.

4. The circular measure of each of the angles of a regular figure of 77 sides is calculated to be 3·06; determine the assumed value of π.

5. Find (i) the number of degrees, (ii) the number of radians, correct to 2 places of decimals, in the angles described by the hands of a clock between 12 noon and 1.25 p.m. on the same day.

6. Find from Tables the value of

$$\cos A \cos B - \sin A \sin B,$$

when $$A = 20^\circ \text{ and } B = 30^\circ.$$

Compare the result with the value of $\cos 50^\circ$.

7. Calculate the value of

$$\sin 30^\circ + \cos^2 30^\circ - \cot^2 45^\circ.$$

8. Prove that

$$\frac{\sec A - \operatorname{cosec} A}{\tan A + \cot A} = \frac{\tan A - \cot A}{\sec A + \operatorname{cosec} A}.$$

[Including Easy Problems and Angles $\ngtr 360^\circ$. Chapters V and VI.]

XVII.

1. How many radians (correct to 2 places of decimals) are there in the following angles: a right angle, the angle of a regular pentagon, of a regular hexagon?

2. Prove that

$$(1 - \cot A)^2 + (1 - \tan A)^2 = (\sec A - \operatorname{cosec} A)^2.$$

3. Draw two lines OA and OB at right angles, making OA 10 cm. long. Draw straight lines AP, AQ, AR cutting OB in P, Q, R, and making with OA angles of 10°, 20°, 40°. Express tan 10°, tan 20°, tan 40° in terms of the lengths on your figure, and by measurement find and write down their values.

Check the accuracy of your result by seeing if

$$\tan 40° = \frac{2 \tan 20°}{1 - \tan^2 20°}.$$

4. At a certain time a lighthouse 23 miles away is seen to be 15° off a ship's course. At what distance, in miles, will the ship pass the lighthouse if she holds on her course? Find the distance by calculation, and also by measurement of a figure drawn to a scale of 1 cm. to a mile.

5. Find from the tables the values of $\cos x$ when $x = 0°$, 10°, 20°, 30°, 40°, 50°, 60°. Draw a curve showing how $\cos x$ varies as x increases from 0° to 60°.

Find from the curve the values of cos 25° and cos 45°, and verify your values by means of the tables.

6. Write down the complements of 30°, $\frac{\pi}{6}$ and the supplements of 47° 57′, $\frac{2\pi}{3}$.

7. If $\sin \alpha = \frac{1}{3}$ and $\cos \beta = \frac{5}{6}$, find the value (correct to 3 places of decimals) of $\sin \alpha \cos \beta + \cos \alpha \sin \beta$.

8. Find the number of radians (correct to 3 places of decimals) in the supplement of the angle of a regular decagon.

XVIII.

1. Prove that

$$\frac{1 + \cos A}{1 - \cos A} = (\operatorname{cosec} A + \cot A)^2.$$

2. Solve the equations

(i) $3 \sin \theta = 2 \cos^2 \theta$,

(ii) $1 - 2 \sin \theta - 2 \cos \theta + \cot \theta = 0$.

3. If a bed of rock dips at an angle of 41° to the horizontal ground, what is its thickness if the section which comes to the surface is 100 metres broad?

4. Prove that

$$\tan^2(180° - A) - \sin^2 A = \{\cos(180° - A) + \sec A\}^2.$$

5. In a triangle ABC, right-angled at C, CE is drawn perpendicular to AB. Prove that, if B = 60°, AE = $\frac{3}{4}$ AB.

6. Find, correct to one second, the time between one and half-past one o'clock when the circular measure of the angle between the hands is $\frac{1}{2}$.

7. If $\sin A = \frac{8}{17}$, find the values of $2 \sin A \cos A$, $\cos^2 A - \sin^2 A$ and $2 \tan A/(1 - \tan^2 A)$, A being acute.

8. The angle C of the triangle ABC is equal to a right angle, and the sides AC, BC are respectively 10 and 20 feet. A perpendicular CD is drawn from C on AB, find the lengths of CD, AD, and BD. (Answer in feet correct to 3 places of decimals.)

XIX.

1. Prove that

(i) $$\frac{\sin A - \cos B}{\sin B - \cos A} = \frac{\cos A + \sin B}{\cos B + \sin A},$$

(ii) $$\sec^3\theta - \cos^3\theta = \tan^2\theta \sec\theta + \sin^2\theta(\sec\theta + \cos\theta).$$

2. Solve the equations

(i) $$\frac{2}{\sqrt{3}} \tan\theta = 2 - \sec^2\theta,$$

(ii) $$\frac{2}{\sqrt{3}} \tan\theta = \sec^2\theta - 2.$$

3. The angle of elevation of the top of a church spire, seen from a distance of 1000 metres, is 5° 12'. Find the height of the spire.

4. Prove that

$$\frac{\sec A + \tan(180° - A)}{\sec A - \tan(180° - A)} - \frac{\tan A - \sec(180° - A)}{\tan A + \sec(180° - A)} = 2 + 4\tan^2 A.$$

5. The angles α and β are acute, $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{5}{13}$. Calculate the value of $\sin\alpha\cos\beta + \cos\alpha\sin\beta$.

6. If the earth's radius is 4000 miles, find the distance from the equator, measured along a line of longitude, of a place whose latitude is 39°.

Also find the distance of the place from the earth's axis.

7. Find from the tables the values of the expression

$$\sin\theta + 3\tan\theta$$

when $\theta = 0°, 10°, 20°, 30°, 40°, 50°, 60°$. Illustrate the variation graphically and thence determine the value of the expression when $\theta = 43°$.

8. If
$$\tan\theta = \frac{b}{\sqrt{a^2 - b^2}},$$
prove that
$$\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta) - \sec\theta = \frac{a}{b}.$$

XX.

1. Two people, 1000 metres apart, standing due South of a balloon, observe the angles of elevation of the balloon to be 18° and 21° 15′ respectively. Find the height of the balloon in metres (correct to $\frac{1}{10}$ of a metre).

2. Prove that

(i) $$\frac{\sin A - \sin(180° - B)}{\cos B + \cos(180° - A)} = \frac{\cos A + \cos B}{\sin B + \sin(180° - A)},$$

(ii) $$(1 + \tan\theta + \sec\theta)^2 + (1 - \tan\theta + \sec\theta)^2 = 4\sec\theta(1 + \sec\theta).$$

3. Solve the equations

(i) $\sqrt{3}(\tan x + \cot x) = 4$,

(ii) $9\sec^4\theta = 16$.

4. An object on the bank of a canal is observed from the opposite side in a direction making an angle of 60° with the bank on which the observer stands. The observer then walks 30 metres along the bank, and finds that the direction of the object makes the same angle of 60° with the bank. Find the breadth of the canal. (Answer in metres, correct to 2 places of decimals.)

5. Given that $\sin A = \frac{20}{29}$, find $\cos A$ and $\cot A$ (i) when A is acute, (ii) obtuse.

6. Prove that

$$\operatorname{cosec}^2 A + \operatorname{cosec}^2 (90° - A) = \operatorname{cosec}^2 A \operatorname{cosec}^2 (90° - A).$$

7. From the tables, calculate the values of $150 \tan 2A$ when $A = 0°, 10°, 20°, 30°, 40°, 50°, 60°, 70°$. Illustrate graphically and thence determine the values of the expression when $A = 33°$ and $62°$.

8. The driving wheel of an engine going 60 miles an hour makes 4 revolutions a second. Find the diameter of the wheel in feet.

XXI.

1. The breaking weight in tons of iron wire rope is equal, roughly, to the square of the circumference in inches; find the value for a rope 3 inches in diameter.

2. Prove that

$$\frac{\operatorname{cosec} A - \sec A}{\operatorname{cosec} A + \sec A}(\cot A - \tan A) = \sec A \operatorname{cosec} A - 2.$$

3. Apply the Tables to find the value of $\frac{\tan \theta}{\theta}$ when θ is the circular measure of $116° 14'$. (Answer to three significant figures.)

4. An arc of a circle of 1200 decimetres radius, subtends at the centre an angle whose circular measure is ·627. Find to 1 centimetre the difference between the length of the arc and the chord joining its extremities.

5. If A, B, C are the angles of a triangle and a, b, c the sides opposite them and AD the perpendicular from A to BC, prove that

$$AD = a/(\cot B + \cot C).$$

If A is obtuse and AD is 7 centimetres and makes angles of 60° and 54° 19′ with AB and AC respectively; calculate the length of BC, correct to 3 places of decimals.

6. Solve the equations

$$\text{(i)} \quad 1 - \cos\theta - \sin\theta + \cot\theta = 0,$$

$$\text{(ii)} \quad \cot^2\theta = 2(1 + 2\sin^2\theta).$$

7. The nautical mile is an arc of the earth's equator which subtends an angle of 1′ at the centre; find its length correct to the nearest foot, using the constants

$$1 \text{ radian} = 206265'',$$

$$\text{earth's equatorial radius} = 20926000 \text{ feet}.$$

8. The current in ampères in a tangent galvanometer is given by the expression $3{\cdot}76 \tan d^\circ$, where d is the deflection.

Illustrate by a graph the connection between the current and deflection, taking for d the values 0°, 5°, 10°, 15°, 20°, 25°, 30°, 35°; from the diagram, determine the value of the current when $d = 13°$. (Answer correct to 2 places of decimals.)

XXII.

1. Prove that

$$\text{(i)} \quad (1 + \tan\theta - \sec\theta)^2 + (1 - \tan\theta - \sec\theta)^2 = 4\sec\theta(\sec\theta - 1),$$

$$\text{(ii)} \quad (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2 = 4\sqrt{\tan^2\theta - \sin^2\theta}.$$

2. Solve the equation

$$3 - 4\cos^2\theta = \tan\theta.$$

3. If $\cos A = \frac{15}{17}$ and A be acute, find the value of

$$\frac{4\cot A + 5\sec A}{3\cos A + 8\sin A}.$$

4. From a window, with his eye 15 decimetres above the roadway, an observer finds that the angle of elevation of the top of a telegraph post is 17° 18′ and that the angle of depression of the foot of the post is 8° 32′. Calculate correct to $\frac{1}{100}$ decimetre the height of the telegraph post and its distance from the observer.

5. There are two routes from A to B. One goes straight from A to B, another goes straight from A to C, and then from C to B. If the perpendicular distance of C from AB is one mile, and if the angle CAB = 32° 40′, and the angle CBA = 45°, find in miles how much longer one route is than the other.

Verify by a diagram drawn roughly to scale.

6. Tabulate the values of $2\sin\theta - \tan\theta$ (correct to 2 places of decimals), when $\theta = 10°, 20°, 30°, 35°, 40°, 50°, 60°$.

By means of a graph find approximately the values of θ, between 0° and 60°, for which the value of this expression $= 0{\cdot}44$.

7. Through how many miles an hour does a certain place move in consequence of the rotation of the Earth? Take the Earth as a sphere of radius 3960 miles and the place to be in latitude 55° 20′. (Answer correct to $\frac{1}{10}$ of a mile.)

8. Show that the area of a road bounded by two concentric circles is the breadth of the road multiplied by the mean between the lengths of the boundaries.

A circular road is 20 metres wide and 1 kilometre long (measured along its central line). Find its area in sq. metres.

[*Including Logarithms and Logarithmic Sines, etc. Chapter VII.*]

XXIII.

1. Find the values of $2{\cdot}307^{0{\cdot}45}$ and $23{\cdot}07^{-1{\cdot}75}$.

2. Two towers A and B on a level plain subtend an angle of 90° at an observer's eye; he walks directly towards B, a distance of 57 yards 2 ft., and then finds that the angle subtended is 121° 5′. Find the distances of A from the two positions of the observer. (Answer correct to $\frac{1}{100}$ of a yard.)

3. With your protractor make an angle XOY of 41°. On OX take A 8 cms. from O and C 14·3 cms. from O, and draw AB and CD perpendicular to OX to meet OY in B and D. On OY take E 13·2 cms. from O and draw EF perpendicular to OY to meet OX in F. Find by measurement and calculation the values of BA/AO, DO/CO, FE/EO.

Write down the value of tan 41°.

4. The index of refraction (μ) is given by $\mu = \sin i/\sin r$. Find μ when the angle of incidence (i) is 21° 28′ and the angle of refraction (r) is 23° 42′.

5. Find the number of radians (correct to 2 places of decimals) in the supplement of the angle of a regular hexagon.

6. If $\theta = 34° 43'$, find its tangent from the tables; then calculate its cosine from the formula $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$ and compare the result with that given in the tables.

7. The distance between two places shown on a map is the horizontal distance, but a surveyor has often to measure up or down a slope. The distance measured in this way is greater than the horizontal distance, and to reduce it to the horizontal distance he multiplies by a number depending on the slope. Make out a table of multipliers for slopes of 5°, 10°, 15°, 20°, 25°.

8. Solve the equation

$$\cot\theta + \tan\theta = 2\operatorname{cosec}\theta.$$

XXIV.

1. A and B are two buoys 800 metres apart, B due N. of A. A vessel passes close to B, and, steering due E., observes that, after 5 minutes, the bearing of A is 33° 27′ south of west. Find, from the tables, the distance the vessel has moved, and check your result by a figure drawn to scale.

2. Find the value of

$$\frac{(91^2 - 37^2) \times 32{\cdot}4}{7 \times 8417}.$$

3. Prove that

(i) $\sin\theta(1+\tan\theta) + \cos\theta(1+\cot\theta) = \sec\theta + \operatorname{cosec}\theta$,

(ii) $\dfrac{1+\sin\theta}{1+\cos\theta} + \dfrac{1-\sin\theta}{1-\cos\theta} = 2\operatorname{cosec}\theta(\operatorname{cosec}\theta - \cos\theta)$.

4. A and B are two points in the same horizontal straight line through the foot C of a tower. The tower subtends angles α and β at A and B respectively. If AB = BC, A being further from the tower than B, prove that

$$\tan\beta = 2\tan\alpha.$$

If $\beta = 23°$, and the height of the tower be 200 ft., find from the tables the value of α and the length of AB.

5. Find the illumination due to a spherical luminary from the formula $\mu\pi \sin^2 \alpha \cos \theta$, where $\mu = \frac{11}{10}$, $\alpha = 8°$, $\theta = 15°$.

6. If $\theta = 17° 15'$, find its tangent from the Tables, then calculate the sine from the formula $\sin \theta = \dfrac{\tan \theta}{\sqrt{1 + \tan^2 \theta}}$. Compare the result with that obtained directly from the Tables.

7. Find from the Tables the variation in the expression $\sin A + 3 \tan A$ when $A = 10°, 20°, 30°, 40°, 50°$. Illustrate graphically and thence determine the value of A for which the expression equals 1·74.

8. If a and b are the sides of a triangle opposite A and B, prove that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$, when $a = 47·54$, $b = 112$, $A = 23°$, $B = 113°$.

XXV.

1. Find the values of the expressions

$$\text{(i)} \quad (5·743)^{1·312},$$
$$\text{(ii)} \quad (\sqrt[8]{·003972}) \times 28·571.$$

2. Prove that

(i) $\operatorname{cosec}^3 \theta + \cot^3 \theta = \operatorname{cosec} \theta \cot \theta (\operatorname{cosec} \theta + \cot \theta) + (\operatorname{cosec} \theta - \cot \theta)$,

(ii) $\operatorname{cosec} A + \sec A + \sin A + \cos A = (\sin A + \cos A)(1 + \operatorname{cosec} A \sec A)$.

3. Find the number of radians (correct to 3 places of decimals) in the supplement of the angle of a regular figure of 21 sides.

4. If a body is just about to slip down a rough inclined plane of angle α, the force required to hold it up is $W \dfrac{\sin (\alpha - \lambda)}{\cos \lambda}$. Find the value of this force when $W = 18·42$ grams-weight, $\alpha = 25° 17'$, λ (= angle of friction) $= 17° 30'$.

5. Find from the tables the values of $\tan 10x - 3 \tan 7x + 1$ when $x = 0°, 1°, 2°, 3°, 4°, 5°, 6°, 7°, 8°, 9°$. Illustrate graphically and thence determine the value of x which makes the expression equal to ·36.

6. The hour-hand of a clock is 26 centimetres long. How many centimetres does its extremity rise (*a*) between 6 and 7 o'clock, (*b*) between 7 and 11 o'clock?

7. Solve the equations:

(i) $\cos\theta = \sin 18^\circ 37' . \cos 137^\circ 14'$,

(ii) $\sin\theta = \sqrt{\sin 10^\circ}$.

8. Find the values of $\tan\theta$, θ, $\sin\theta$, where θ is the circular measure of an angle of (i) 32°, (ii) 65° 15′, and thus show that

$$\tan\theta > \theta > \sin\theta.$$

XXVI.

1. The distance of the Centre of Gravity of a segment of a circle from the centre of the circle is $\frac{4}{3} r \frac{\sin^3 a}{2a - \sin 2a}$, where r is the radius and $2a$ the angle subtended at the centre.

Find the value when $r = 30$ cms., $a =$ circular measure of 18°.

2. Prove that

(i) $(1 - \sin\theta)(\tan\theta + \sec\theta) = \cot\theta (1 - \cos\theta)(\cot\theta + \operatorname{cosec}\theta)$,

(ii) $\sin\theta (1 - \tan\theta) + \cos\theta (\cot\theta - 1) = \operatorname{cosec}\theta - \sec\theta$.

3. Find the values of $5{\cdot}607^{1\cdot3}$ and $56{\cdot}07^{-1\cdot25}$.

4. Look out the values of sin 24° and cos 24° from the Tables, then write down the sine and cosine of 156°, of 204° and of 336°; also write down the sine and cosine of 114° and find to 3 places of decimals the tangent of 114°.

5. If the radius of a circle is 127 decimetres, find the arc which subtends 33° 12′ at the centre. (Answer to 1 millimetre.)

6. Solve the equations:

(i) $\sin x + \operatorname{cosec} x = \frac{34}{15}$,

(ii) $3\tan^2 x - 4\tan x + 1 = 0$.

7. From certain experiments Young's Modulus is found to be $\dfrac{4536 \times 981\cdot3 \times 271}{\cdot133 \times (\cdot05334)^2 \times 3\cdot1416}$. Calculate the value of this expression.

8. Find the values of $1 - \cos 2\theta$ when $\theta = 0°, 5°, 10°, 15°, 20°, 25°$. Illustrate graphically and thence determine the value of θ when $1 - \cos 2\theta$ equals $0\cdot293$.

XXVII.

1. Find x from the equation

$$(1\cdot235)^x = (6\cdot543)^{\frac{1}{x}}.$$

2. Find the value of

$$\sqrt[5]{2\cdot709} \times \sqrt[7]{1\cdot2387}.$$

3. Prove that

(i) $\dfrac{1 - \cos\theta}{1 - \operatorname{cosec}\theta} - \dfrac{1 + \cos\theta}{1 + \operatorname{cosec}\theta} = 2\tan\theta(\sin\theta - \sec\theta)$,

(ii) $(\sec A - 2\sin A)(\operatorname{cosec} A + 2\cos A)\sin A\cos A$
$$= (\cos^2 A - \sin^2 A)^2.$$

4. Two posts of the same height stand on either side of a road 120 ft. wide; at a point in the road between the posts the elevations of the tops of the pillars are 57° 30′ and 32° 30′. Find the height of the posts and the position of the point.

5. Prove that

$$\cos(360° - A) + \sin(270° + A) + \cos(180° - A) - \sin(270° - A) = 0.$$

6. If θ is the number of radians in 42°, prove that

$$\sin\theta > \theta - \frac{\theta^3}{4} \quad \text{and} \quad \cos\theta > 1 - \frac{\theta^2}{2}.$$

7. The perimeter of a sector of a circle is 15 metres, and the radius of the circle 4 metres; find the angle of the sector to the nearest second.

8. Solve the equation

$$3(\sec^2 x + \tan^2 x) = 5.$$

XXVIII.

1. Prove that

$$(1 - \cos\theta)(\cot\theta + \operatorname{cosec}\theta) = \tan\theta(1 - \sin\theta)(\tan\theta + \sec\theta).$$

2. Find the value of $\sqrt{a^2 + b^2}$, when $a = 713\cdot5$ and $b = 42\cdot87$.

3. Evaluate the expression:

$$\frac{\sin 56^\circ 24' \,.\, \sin 23^\circ 31' \,.\, \sin 42^\circ 19'}{\sin 82^\circ 52' \,.\, \sin 49^\circ 43' \,.\, \sin 71^\circ 15'}.$$

4. If a strip of paper 1 kilometre long and ·002 decimetre thick, is rolled up into a solid cylinder, find the radius of the circular ends of the cylinder to the nearest millimetre.

5. What is the angle of elevation of the sun when the length of the shadow of a pillar is 5 times the height of the pillar?

6. If $\cos\theta = \frac{20}{101}$, find $\cot(90^\circ + \theta)$.

7. Given that the radius of the earth is 4000 miles, what is the latitude of a place distant 2510 miles from the earth's axis, and what is its distance, to the nearest mile, measured along a meridian, from the equator?

8. Find from the Tables the logarithms of 200, 201, 202, 203, 204. Represent on squared paper the increments of the logarithms corresponding to the addition of 1, 2, 3, 4 to the number 200. Show how to find, with the help of the diagram, log 202·2, log 202·4. Compare the result with that given by the table of differences.

[*Including Elementary Properties and Solution of Triangles. Chapters VIII and IX.*]

XXIX.

1. Prove that

$$\sec^2 A + \tan^2 A = \sec A \tan A (\sec A + \tan A) + (\sec A - \tan A).$$

2. Prove that

$$a^2 + b^2 + c^2 = 2(ab \cos C + bc \cos A + ca \cos B).$$

3. Calculate the angles B and C, given $b=15$, $c=12$ and $A=37° 48'$. [Verify by drawing a figure to scale.]

4. Find the values of

$$\text{(i)}\quad (8{\cdot}417)^{3{\cdot}142},$$

$$\text{(ii)}\quad \frac{925{\cdot}7\times 82{\cdot}3\times 101{\cdot}9}{54{\cdot}73}.$$

5. If the three sides of a triangle are 15, 17·22 and 14·9 centimetres respectively, find the angle opposite the greatest side.

6. Given that the base BC of a triangle measures 8 centimetres and $BA=5$ cms., calculate the area of the triangle ABC, when $A\hat{B}C=0°, 20°, 40°, 60°, 80°, 100°, 120°, 140°, 160°, 180°$, respectively. Illustrate graphically and thence deduce the value of the angle when the triangle has its maximum area.

7. The four angles of a quadrilateral are in A.P., and the difference of the greatest and least is equal to a right angle. Express each of the four angles in degrees and also in circular measure, correct to 3 places of decimals. ($\pi=3{\cdot}1416$.)

XXX.

1. Prove that

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$$

2. The elevation of a tower from a point A due N. of it is observed to be 45°, and from a point B due E. of it to be 32°. If $AB=230$ feet, find the height of the tower.

3. Solve the equation

$$4\sin^2\theta+3\operatorname{cosec}^2\theta=7.$$

4. Prove that

$$\sec^6 A-1=\tan^2 A\,(\tan^4 A+3\tan^2 A+3).$$

5. Find x from the equation $(\frac{10}{3})^{x+2}=9^{2x-1}$.

6. Given that $a=17{\cdot}2$, $b=16{\cdot}5$, $c=14{\cdot}3$, find the value of C. Check by a figure drawn approximately to scale.

7. If $a=1021$ cms., $b=723$ cms. and $B=41°$, find A.

XXXI.

1. Solve the equation
$$2 \sin^2 \theta + 3 \cos \theta - 3 = 0.$$

2. Prove that
$$\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2}.$$

3. If the area of a sector of a circle whose angle is 8° is 13 sq. centimetres, find the circumference of the circle.

4. Do the solutions of the following triangles give any ambiguity?

(i) $a = 15$, $b = 21\cdot2$, $A = 31°$,

(ii) $a = 5\cdot2$, $b = 4\cdot1$, $A = 58°$,

(iii) $a = 3\cdot9$, $b = 4\cdot21$, $A = 62° \, 30'$.

5. Find the value of

(i) $\dfrac{82\cdot74 \times 72\cdot31 \times (7\cdot41)^2}{(9\cdot234)^3}$,

(ii) $(82\cdot41)^{0\cdot715}$.

6. If $c = 82\cdot97$, $a = 41\cdot35$ and $B = 41° \, 22'$, find the values of A, C and b.

7. O is the centre of the circle inscribed in a sector of a circle whose angle is 60°. From O the lines OD and OF are drawn at right angles to the bounding radii of the sector. Find the ratio of the area of the given sector to that of the smaller sector thus formed in the inscribed circle.

XXXII.

1. In any triangle prove that
$$\tan A = \frac{a \sin B}{c - a \cos B}.$$

2. A man sees a cairn on the edge of a cliff and observes that its angle of elevation is 30°. He walks 234·24 feet on level ground straight towards it, and finds its elevation now to be 45°. What is its height above him? How much nearer (to $\frac{1}{10}$ of a foot) must he walk on the level to make the elevation of the cairn increase to 60°? ($\sqrt{3} = 1{\cdot}732$.)

3. If ABC is a triangle in which the angle B is 58°, the angle C is 39° 12′, and the perpendicular AD drawn from the angle A to the side BC is 15 centimetres long, find the lengths of the three sides of the triangle ABC.

4. If the volume of a sphere is $\frac{4}{3}\pi r^3$, find r the radius when the volume is 127 cu. centimetres. ($\pi = 3{\cdot}142$.)

5. If the mean distance of the earth from the sun is 92·9 millions of miles, and its time of revolution 365·3 days, how many miles a second does the earth travel? ($\pi = 3{\cdot}142$.)

6. Given that $a = 35{\cdot}27$ cms., $b = 14{\cdot}95$ cms. and C = 53° 42′, find the values of A, B and c.

7. Prove that

$$\operatorname{cosec}^4 A\,(1 - \cos^4 A) - 2\cot^2 A = 1.$$

XXXIII.

1. Prove that

$$(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c.$$

2. Find the area of a triangle the sides of which are 5·1, 7·83, 4·97 centimetres respectively.

3. From a point A, a church bears N. 14° E. and a tree E. 13° N. From the tree, the church bears N. 13° W. The distance from A to the tree is 5 kilometres. Find the distance between A and the church.

4. Find the value of

$$\sqrt[5]{\frac{7800 \times {\cdot}00167 \times 42{\cdot}9}{\frac{1}{2}(3152)^{\frac{1}{3}}}}.$$

5. If A = 27° 15′, $b = 126{\cdot}9$, $a = 83{\cdot}24$, find B.

6. At what approximate distance must a coin, 2 centimetres in diameter, be placed from a man, in order that the sun may just be hidden; the angle subtended by the sun's diameter being 32′?

7. Solve the equation

$$\cos\theta - \sqrt{3}\sin\theta = 1.$$

XXXIV.

1. If $\dfrac{9^x}{3^{x+y}} = 27$ and $2x = 5y$, find x and y.

2. Solve the equation

$$(810)^{x+1} = 7 \times (70\cdot56)^x.$$

3. If the sides of a triangle are 14, 16 and 18 centimetres respectively, find the angles opposite the greatest and smallest sides. Check by a figure drawn to scale.

4. In the face of a vertical cliff, a mark 50 metres above its base has an altitude of 32° as observed at a point on a level with the base of the cliff. At the same point the altitude of the top of the flagstaff on the summit of the cliff, directly over the mark, is observed to be 45°. Find the height of the top of the flagstaff above the mark.

5. Find the side of a square inscribed in a circle of circumference 5 metres. ($\pi = 3\cdot142$.)

6. Prove that

$$b(b+c-a)(1-\cos A) = a(a+c-b)(1-\cos B).$$

7. Calculate the values of $\sin 2A - 2\sin A - 1$ when A has the values 0°, 5°, 10°, 15°, 20°, 25°. Illustrate graphically, and thence determine the value of A for which the expression equals $-1\cdot05$.

[*Including Heights and Distances, using logarithms. Chapter X.*]

XXXV.

1. A flagstaff 67 decimetres high, standing on the edge of a cliff, subtends an angle of $0^\circ\ 42'$ at a ship at sea, the angle of elevation of the cliff being 15°. Find the distance (in metres) of the base of the cliff from the ship.

2. In a triangle ABC, if $a = 35{\cdot}47$ cms., $b = 26{\cdot}21$ cms. and $B = 38^\circ\ 12'$, find the two values of c.

Check by the formula $c_1 c_2 = a^2 - b^2$.

3. If three circles of radius 3·5 centimetres touch one another, find the area between them. ($\pi = 3{\cdot}142$.)

4. Find the value of

$$\sqrt[3]{{\cdot}0002471} \times 82{\cdot}95.$$

5. Prove that

$$a \cos B - b \cos A = c \operatorname{cosec}^2 C (\sin^2 A - \sin^2 B).$$

6. In a quadrant of a circle another circle is inscribed. Prove that its area is $\dfrac{1}{3 + 2\sqrt{2}}$ of the area of the first circle.

7. A bed of coal 3·2 metres thick is inclined at 22° to the surface. Calculate the number of kilograms of coal that lie under 5000 sq. metres of surface. 1016 kilograms of coal occupy 793 cu. decimetres. (The 3·2 metres is to be regarded as a measurement at right angles to the surface of the coal bed.)

XXXVI.

1. A man 1·75 metres high standing 39·62 metres from the foot of a tower observes the elevation of the tower to be $30^\circ\ 14'$. Find the height of the tower.

2. In a triangle ABC the angles B and C are equal, and the tangent of each of these angles is $\frac{4}{3}$. Determine the value of the third angle by means of tables and then verify by the construction of a figure.

3. Calculate the value of $\sqrt{b^2+c^2-2bc\cos A}$ when $b=123{\cdot}6$, $c=41{\cdot}23$, $A=40°\ 52'$ given that the expression equals $(b+c)\cos\phi$, where

$$\sin\phi=\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}.$$

4. From the masthead of a ship, a rock is seen under an angle of depression of $4°\ 41'$, while another rock, 150 metres away from the ship in the same direction, is seen under an angle of depression of $9°\ 12'$. Find the distance between the rocks.

5. Prove that

$$\frac{b^2-c^2}{a}\cos A+\frac{c^2-a^2}{b}\cos B+\frac{a^2-b^2}{c}\cos C=0.$$

6. If $a=17{\cdot}24$, $b=15{\cdot}48$ and $C=29°\ 14'$, find A and B.

7. Find approximately, in minutes, the inclination to the horizon of an incline which rises 1·07 metres in 225 metres.

XXXVII.

1. A spherical glass vessel has a cylindrical neck 8 cm. long, 2 cm. diameter; the diameter of the spherical part is 8·5 cm. By measuring the amount of water it holds, a boy makes out its volume to be 345 cu. cm. Find by calculation whether he is correct, taking the above as inside measurement and π as 3·142. Take the spherical vessel and the cylindrical neck to be complete, neglecting the fact that they overlap.

2. Calculate the values of

$$\text{(i)}\quad \frac{72{\cdot}41\times 373{\cdot}9\times{\cdot}0257}{82{\cdot}47\times{\cdot}5891},$$

$$\text{(ii)}\quad (1{\cdot}425)^{4{\cdot}271}.$$

3. In a triangle ABC, $a=94$ cms., $B=58°\ 21'$, $C=42°\ 14'$. Calculate the length of c.

Also find what error is made in the length of c if the angle C is through a wrong measurement taken as $42°\ 17'$.

4. Two spectators, at two stations 32 metres apart, observe, at the same instant, the altitude of a kite, and find it to be $38° 18'$ at each place. The angle which the line joining one station and the kite subtends at the second station is $57° 14'$. Find the height of the kite at the moment of observation.

5. Find the ratio of an angle of a regular polygon of $2n$ sides to an angle of a regular polygon of n sides. Check your result by supposing $n=4$.

6. Tangents are drawn to a circle of radius 1 centimetre from a point distant 3 cms. from the centre. Find the length of the chord joining the points of contact, and prove that the area of the triangle contained by it and the tangents is approximately 2·5 sq. cms.

7. Prove that

$$(1+\sin A)^2\{\cot A+2\sec A(1-\operatorname{cosec} A)\}+\operatorname{cosec} A\cos^3 A=0.$$

XXXVIII.

1. Draw the graph of $2\sin\theta+3\cos\theta$ for values of θ between $0°$ and $180°$, and from your figure state the greatest value of the expression between those limits. For what values of θ is the expression equal to 2·5?

2. A and B are the summits of two mountains which rise from a horizontal plain, B being 1000 ft. above the plain. Find the height of A; it being given that its angle of elevation, as seen from a point C in the plain (in the same vertical plane with A and B), is $50°$; while the angle of depression of C, viewed from B, is $28° 58'$, and the angle subtended at B by AC is $50°$.

3. Solve the equation

$$6\tan^2\theta-4\sin^2\theta=1.$$

4. Prove that

$$a(\cos B\cos C+\cos A)=b(\cos C\cos A+\cos B)$$
$$=c(\cos A\cos B+\cos C).$$

5. Prove that

$$2\sec^2 A-\sec^4 A-2\operatorname{cosec}^2 A+\operatorname{cosec}^4 A=\cot^4 A-\tan^4 A.$$

6. Solve the equation

$$(723)^{2x+1}=8\times(829)^{x-1}.$$

7. If $a=826·1$, $b=741·5$ and $B=42° 15'$, find A.

XXXIX.

1. Solve the equations

(i) $10^{x-1} = 2{\cdot}351$,

(ii) $\tan x = \sin 67° \, 30' \, . \cot 17° \, 14'$.

2. The mutual distances of three points in a horizontal plane, from which the elevations of an inaccessible object are the same, are 732, 820 and 924 metres. Find the height of the object, its elevation from each of the three stations being 36°.

3. Prove that

$$\cos A \, (2 \sec A + \tan A) \, (\sec A - 2 \tan A) = 2 \cos A - 3 \tan A.$$

4. Two places on the same meridian are 192·5 kilometres apart; find their difference in latitude, the earth's diameter being 12,700 kilometres. Answer to the nearest minute. $(\pi = 3{\cdot}1416.)$

5. In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are 2 and 3.

6. A person in a ship under sail sees two objects known to be 5 miles asunder, the one N.N.E., the other N.E.; after sailing an hour and ten minutes due East, he sees the same objects in a right line, and $7\frac{1}{2}°$ W. of N. Find the rate of sailing and the distance of the ship from the nearest object at the time of the last observation.

7. Find the value of Young's Modulus from the formula

$$\frac{4536 \times 981{\cdot}3 \times 271}{{\cdot}133 \times ({\cdot}05334)^2 \times 3{\cdot}142}.$$

[*Including Functions of Compound Angles. Chapters XI and XII.*]

XL.

1. Prove

(i) $$\tan \frac{A}{2} \cot \frac{B}{2} - \cot \frac{A}{2} \tan \frac{B}{2} = \frac{2 (\cos B - \cos A)}{\sin A \sin B},$$

(ii) $$\frac{2 \sin 2A - \sin 3A}{\cos A} = \sin A \left(8 \sin^2 \frac{A}{2} + \sec A\right).$$

2. A tower stands on a horizontal plane, from two points on the plane distant 15 metres from its base respectively. The angle of elevation in the former case is three times that in the height of the tower.

3. In solving a triangle when given a, b, A, C are C_1 and C_2, show that

$$\sin \frac{C_1 + C_2}{2} = \cos A.$$

4. If $A + B + C = 180°$, show that

$$\frac{\sin B + \sin C - \sin A}{\sin B + \sin C + \sin A} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

5. The difference between two angles is 1 measure of their sum is 1; find them in degrees

6. Eliminate y between the equations

$$\sqrt{3} \tan x + \tan y = \sqrt{3} + 1; \quad x + y$$

7. If $$\sin 2\beta = \frac{\sin 2a + \sin 2a'}{1 + \sin 2a \sin 2a'},$$

prove $$\tan\left(\frac{\pi}{4} + \beta\right) = \pm \tan\left(\frac{\pi}{4} + a\right) \tan\left(\frac{\pi}{4}\right.$$

XLI.

1. Given
$$\sin 21° 20' = \cdot 3638,$$
$$\cos 21° 20' = \cdot 9315,$$

find A, B and C (all less than four right angles)

$$\sin A = -\cdot 3638 \text{ and } \cos A = -\cdot 931$$
$$\sin B = -\cdot 9315 \text{ and } \cos B = -\cdot 363$$
$$\sin C = -\cdot 3638 \text{ and } \cos C = +\cdot 931$$

2. ABC is a right-angled triangle, and the the right angle C cuts AB in D, show that

$$DB = AD \tan A.$$

3. The hypotenuse of a right-angled triangle is 1000 metres long and the difference between the other two sides is 240 metres; calculate the other sides of the triangle, and check your result by drawing the triangle to scale.

4. Prove that

(i) $\sec^2 A (1 + \sec 2A) = 2 \sec 2A$,

(ii) $\sin (x + z) \sin y - \sin (y + z) \sin x = \sin z \sin (y - x)$.

5. Prove that

$$\sin 7A = (1 + 2 \cos 2A + 2 \cos 4A + 2 \cos 6A) \sin A.$$

6. Solve the equation

$$5 \log x + 3 \log \frac{x}{2} = \log 2592.$$

7. Find the values of $\tan \theta$ from the equation

$$(n + 1) \sin 2\theta + (n - 1) \cos 2\theta = n + 1.$$

XLII.

1. Prove that

$$8 (\sin^2 42^\circ - \cos^2 78^\circ) = \sqrt{5} + 1.$$

2. If θ is the smaller of the two angles into which a given angle A is divided, and if the sines of the two parts are in the ratio of 4 to 5, show that

$$9 \tan \left(\frac{A}{2} - \theta\right) = \tan \frac{A}{2}.$$

3. Prove that

(i) $(\sin A - \cos A)^4 + (\sin A + \cos A)^4 = 3 - \cos 4A$,

(ii) $(\cot^2 A - \tan^2 A) = \dfrac{8 \cos 2A}{1 - \cos 4A}$.

4. ABC is a triangle such that if the straight line AD be drawn within the angle A making the angle BAD double the angle DAC, this line will intersect BC in D so that

$$BD = 2DC;$$

show that $$b = c \cos CAD,$$

and $$a^2c^2 = (c^2 - b^2)(c^2 + 8b^2).$$

5. Find the positive values of θ less than $180°$ w
the equations

$$\text{(i)}\quad 17\sin\theta = 15\sin 63°\,18',$$

$$\text{(ii)}\quad \cos\theta = \cos 37°\,59'\cos 153°\,18'.$$

6. Show that

(i) $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\right.$

(ii) $\cos\frac{\pi}{16}+\cos\frac{3\pi}{16}+\cos\frac{5\pi}{16}+\cos\frac{7\pi}{16}=\frac{1}{2}\operatorname{cosec}\frac{11}{1}$

7. Show that

(i) $\sin x+\sin y+\sin z-\sin(x+y+z)$

$$=4\sin\frac{y+z}{2}\sin\frac{z+x}{2}$$

(ii) Express in factors

$$\sin 2n\mathrm{A}+\sin 2n\mathrm{B}+\sin 2n\mathrm{C},$$

where n is an integer and

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=\pi.$$

XLIII.

1. Prove that

(i) $2\sin 5\mathrm{A}-\sin 3\mathrm{A}-3\sin\mathrm{A}=4\sin\mathrm{A}\cos^2\mathrm{A}\,(1$

(ii) $\tan\alpha+\tan\left(\frac{\pi}{3}+\alpha\right)+\tan\left(\frac{2\pi}{3}+\alpha\right)=3\tan$

2. If $\mathrm{A}+\mathrm{B}+\mathrm{C}=\pi$, prove that

$\sin^3\mathrm{A}+\sin^3\mathrm{B}+\sin^3\mathrm{C}$

$$=3\cos\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}+\cos\frac{3\mathrm{A}}{2}\cos\frac{3\mathrm{B}}{2}$$

3. Solve the equations

(i) $x^2-\sqrt{2}\sin\left(\frac{\pi}{4}+\alpha\right)x+\frac{1}{2}\sin 2\alpha=0.$

(ii) $x^2-2\cot 2\beta\,.\,x-1=0.$

4. P and Q are two stations 1000 metres apart on a straight [illegible]tch of sea shore and P is due East of Q. At P a rock bears [illegible] W. of S., at Q it bears 35° E. of S. Find the distance of the [illegible]k from the shore to the nearest metre.

5. A, B, C, D are consecutive angular points of a regular [illegible]agon, show that

$$AB : AC : AD :: \sqrt{2-\sqrt{2}} : \sqrt{2} : \sqrt{2+\sqrt{2}}.$$

6. ABD is a triangle whose sides BD, DA, AB are 3, 4, 5 cms. [illegible]pectively; BCD is an equilateral triangle and A and C are on [illegible]posite sides of BD; show that

$$\sin ACD = \frac{2}{\sqrt{25+12\sqrt{3}}}.$$

7. Given $\tan\theta = \dfrac{\tan\alpha + \tan\beta}{1+\tan\alpha\tan\beta}$,

[illegible]ve that $\sin 2\theta = \dfrac{\sin 2\alpha + \sin 2\beta}{1+\sin 2\alpha \sin 2\beta}$.

XLIV.

1. Prove that

(i) $\sin A + \sin 5A + \sin 9A - \sin 15A = 4 \sin 3A \sin 5A \sin 7A$,

(ii) $\operatorname{cosec} A + \operatorname{cosec}\left(A + \dfrac{2\pi}{3}\right) + \operatorname{cosec}\left(A + \dfrac{4\pi}{3}\right) = 3 \operatorname{cosec} 3A$.

2. Prove that

$$\frac{\cos 9\theta}{\cos 3\theta} - \frac{\cos 18\theta}{\cos 6\theta} = 2\{\cos 6\theta - \cos 12\theta\}.$$

3. To find the breadth AB of a river an observer measures AB produced a length BC of 20 metres and then walks a [illegible]stance CP of 100 metres at right angles to AC. He finds that [illegible] subtends 35° 40′ at P. Find the breadth of the river and the [illegible]gle BC subtends at P.

4. Prove that

(i) $[\sin\theta(1+\sin\theta) + \cos\theta(1+\cos\theta)]$
$[\sin\theta(1-\sin\theta) + \cos\theta(1-\cos\theta)] - 2\sin\theta\cos\theta = 0$,

(ii) $(1+2\cos\theta\tan\theta)(2-\sec\theta\cot\theta) = \cos\theta(3\tan\theta - \cot\theta)$.

5. In a triangle the parts a, b, A are given and $b > a$, prove that if c_1, c_2 are the two values of the third side

$$c_1 c_2 = b^2 - a^2.$$

6. D is the middle point of AB, the common base of three isosceles triangles, whose vertices are C_1, C_2, C_3, also

$$2C_1D = C_2D = AB \text{ and } 2C_3D = 3AB.$$

Show that the three vertical angles are together equal to two right angles.

7. M is the middle point of the side BC of a triangle ABC, which is such that 3AC = 4AB; and 2AM = 3BC; show that

$$\tan \frac{AMB}{2} = \sqrt{\frac{4}{11}}.$$

XLV.

1. Prove that

(i) $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$,

(ii) $\sec \theta - \tan \theta = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$.

2. Points A, B, C, D are taken on the circumference of a circle so that the arcs AB, BC, CD subtend respectively at the centre angles of 108°, 60°, 36°, show that

$$AB = BC + CD.$$

3. Prove that

(i) $\sin\frac{x-y-z}{2} \sin\frac{y-z}{2} + \sin\frac{x+y-z}{2} \sin\frac{y+z}{2} = \sin\frac{1}{2}x \sin y$,

(ii) if $A + 2B = 180°$,

$$2\sin^2(A - B)(2 - \cos A) = (\sin^2 A + 2\sin^2 B)(1 - 8\cos A \cos^2 B).$$

4. Find $\cos\theta$ from the equation

$$\{4\cos(\theta + \alpha) - 1\}\{4\cos(\theta - \alpha) - 1\} = 5(2\cos 2\alpha - 1).$$

5. Prove that

(i) $8\sin(A+45^\circ)\sin(B+45^\circ)\sin(A-45^\circ)\sin(B-45^\circ)$
$$=\cos(2A+2B)+\cos(2A-2B),$$

(ii) $\operatorname{cosec} A + 2\operatorname{cosec} 2A = \sec A \cot\dfrac{A}{2}$.

6. In a triangle ABC the angle A is x degrees, the angle B, $10x$ grades, and the circular measure of C is $\dfrac{\pi x}{9}$. Find the number of degrees in each angle.

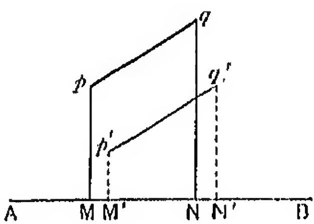
7. Prove that

(i) $\cos\theta - 2\cos 5\theta + \cos 9\theta = 2\sin 2\theta(\sin 3\theta - \sin 7\theta)$,

(ii) $\sin^3\theta\sin 3\theta + \cos^3\theta\cos 3\theta = \cos^3 2\theta$.

APPENDIX ON PROJECTION.

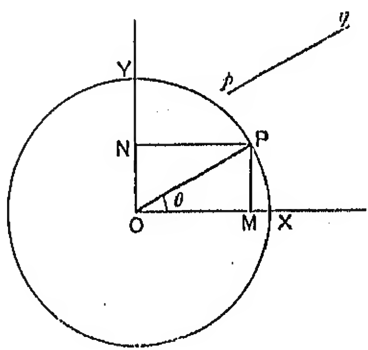
I. Def. If from p and q, the extremities of a line pq, perpendiculars pM, qN be drawn to another line AB, then MN *is called the projection* of the line pq on the line AB.



If $p'q'$ is equal and parallel to pq, then its projection M′N′ is obviously equal to MN, the projection of pq.

Theorem I. *To find the length of the projection of a line pq on* OX.

Draw OP parallel and equal to pq and PM perpendicular to OX. With centre O and radius OP describe a circle.



The projection of pq on OX

$$= \text{proj. of OP on OX}$$

$$= \text{OM}$$

$$= \text{OP} \cos \theta,$$

where θ is the angle obtained by rotating in a *positive direction* from OX to OP.

If OY is perpendicular to OX,

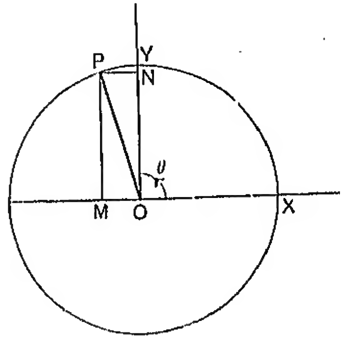
$$\text{the projection of OP on OY} = \text{ON}$$

$$= \text{MP} = \text{OP} \sin \theta.$$

If the angle θ is in the second quadrant,

proj. of OP on OX $=$ OM

$= \text{OP} \cos \theta,$

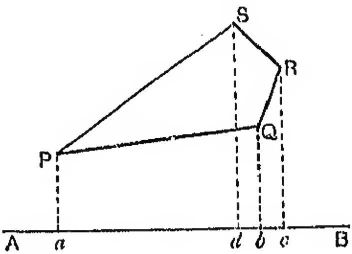


and similarly for the third and fourth quadrants. It is thus obvious that the projection of OP on OX will be negative in the second and third quadrants. It may similarly be shown that the projection of OP on OY, the line at right angles to OX, is always OP $\sin \theta$.

Thus the projection of a line pq on OX in all cases equals

$pq \times$ *cosine (angle between pq and the positive direction of* OX).

Theorem II. If the sides of a rectilineal figure PQRS be projected on a line AB, then



projection of PQ $= ab$,

" " QR $= bc$,

" " RS $= cd = -dc$,

" " PS $= ad$.

∴ sum of projections of

PQ, QR, RS $= ab + bc - dc$

$= ac - dc$

$= ad$

$=$ projection of PS.

It is thus also at once obvious that the sum of the projections of PQ, QR, RS, SP $=$ projection of PS $+$ projection of SP

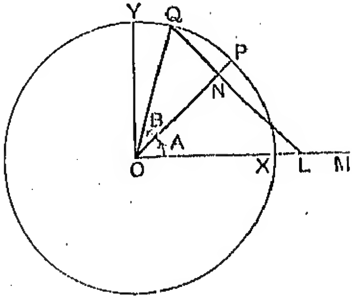
$= ad + da = ad - ad = 0.$

∴ (i) *The sum of the projections on any line of any number of broken lines joining two points* P, S $=$ *projection of* PS *on the same line.*

(ii) *The sum of the projections of the sides of a polygon on any straight line is zero.*

The Addition Formulae.

Method I.



Take a line OX = unit of length.

With centre O and radius OX describe a circle.

Let a line starting in the position OX rotate through an angle A to the position OP, and then through a further angle B to the position OQ.

From Q draw QN perp. to OP.

Produce QN to meet OX produced in L.

Angle between OX and OQ = A + B.

" " OX and ON = A.

" " OX and NQ = MLQ

= ONL + LON

= 90° + A.

Now proj. of OQ on OX = OQ cos (A + B) = cos (A + B), since OQ = 1,

proj. of ON on OX = ON cos A = OQ cos B cos A

= cos B cos A,

proj. of NQ on OX = NQ cos (90° + A) = − NQ sin A

= − OQ sin B sin A = − sin B sin A,

and the projection of OQ on OX = sum of projections of ON and NQ; (Theorem II.)

$\therefore$ cos (A + B) = cos A cos B − sin A sin B.

If OY is perp. to OX, then

projection of OQ on OY = OQ sin (A + B) = sin (A + B),

projection of ON on OY = ON sin A = OQ cos B sin A

= cos B sin A,

projection of NQ on OY = NQ sin (90° + A) = NQ cos A

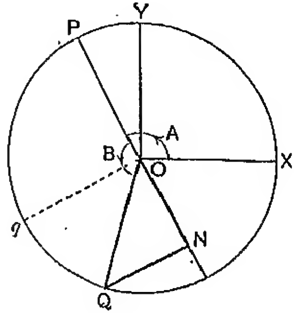
= OQ sin B cos A = sin B cos A,

and projection of OQ on OY = sum of projections of ON and NQ;

$\therefore$ sin (A + B) = sin A cos B + cos A sin B.

If A *and* B *are both obtuse,*

From Q draw a perpendicular QN on PO produced, and Oq parallel to NQ.



Angle between OX and OQ

$= A + B.$

Angle between OX and ON

$= A + 180°.$

Angle between OX and NQ

= angle between OX and Oq

$= A + 90°.$

$\therefore$ projection of OQ on OX $= OQ \cos (A+B) = \cos (A+B)$,

projection of ON on OX $= ON \cos (A + 180°) = - ON \cos A$

$= - OQ \cos (180° - B) \cos A$

$= \cos B \cos A$,

projection of NQ on OX $= NQ \cos (A + 90°)$

$= - NQ \sin A$

$= - OQ \sin (180° - B) \sin A$

$= - \sin B \sin A$,

and projection of OQ on OX = sum of projections of ON and NQ,

$\therefore \cos (A+B) = \cos A \cos B - \sin A \sin B.$

If OY is perpendicular to OX, then

projection of OQ on OY $= OQ \sin (A+B) = \sin (A+B)$,

projection of ON on OY $= ON \sin (A + 180°) = - ON \sin A$

$= - OQ \cos (180° - B) \sin A$

$= \cos B \sin A$;

projection of NQ on OY $= NQ \sin (A + 90°)$

$= NQ \cos A$

$= OQ \sin (180° - B) \cos A$

$= \sin B \cos A$,

and projection of OQ on OY = sum of projections of ON and NQ;

$\therefore \sin (A+B) = \sin A \cos B + \cos A \sin B.$

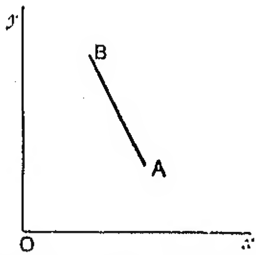
Similar proofs may be obtained when A and B have other magnitudes.

In both these formulae, writing $-$ B for B, we have

$$\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B),$$
$$\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B);$$
$$\therefore\ \cos(A-B)=\cos A\cos B+\sin A\sin B,$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B.$$

II. *Alternative Method.*

Let Ox and Oy be two rectangular axes and let the direction AB make an angle θ (where θ is the angle Ox must rotate through in order that it may be parallel to AB) with Ox.



Then a point moving along the straight line AB in either direction will be said to move in a direction θ with respect to the axes Ox, Oy.

If it moves in a direction from A to B it will be considered to have moved a *positive* distance, but if it moves in the direction from B to A, it will be considered to move a *negative distance.*

Theorem I. *If* Ox *and* Oy *are two rectangular axes, and if a point moves from* O *to* P *a distance* r *units in the direction* θ, *the coordinates of* P *are*

$$r\cos\theta \text{ and } r\sin\theta.$$

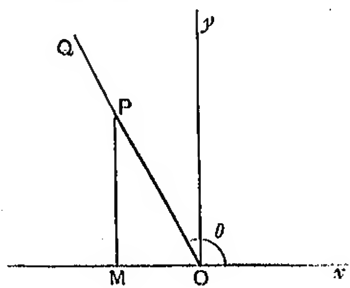
Let OQ be in the direction θ and let x and y be the coordinates of P.

From P draw PM perpendicular to Ox.

Then, no matter in which quadrants OQ and P lie, we always have $OM=x$ and $MP=y$, both in magnitude and sign.

Case I. *Let r be positive.*

Then, no matter in which quadrant P and Q lie, we have by the definitions of $\sin\theta$ and $\cos\theta$,



$$\cos\theta=\frac{OM}{OP}=\frac{x}{r},$$
$$\sin\theta=\frac{MP}{OP}=\frac{y}{r};$$
$$\therefore\ x=r\cos\theta,$$
$$y=r\sin\theta.$$

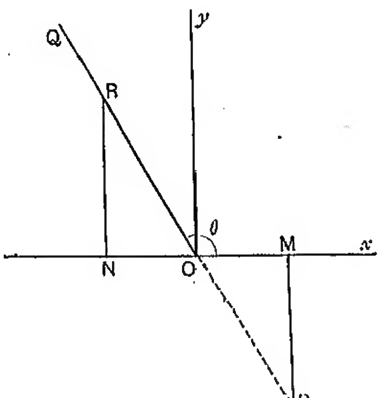
CASE II. *Let r be negative.*

Along the line OQ, cut off OR equal but opposite in sign to OP, and draw RN perpendicular to Ox. Then, no matter in which quadrants P, Q and R lie, we have

$$OR = -OP,$$
$$ON = -OM,$$
$$NR = -MP;$$

therefore by the definitions of $\sin\theta$ and $\cos\theta$,

$$\cos\theta = \frac{ON}{OR} = \frac{-OM}{-OP} = \frac{OM}{OP} = \frac{x}{r},$$
$$\sin\theta = \frac{NR}{OR} = \frac{-MP}{-OP} = \frac{MP}{OP} = \frac{y}{r};$$
$$\therefore\ x = r\cos\theta,$$
$$y = r\sin\theta.$$



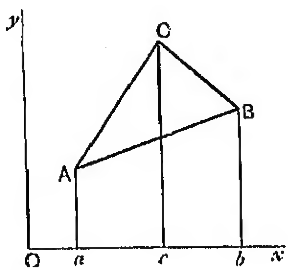
We have thus shown that the theorem is true for all values of r and θ, positive or negative.

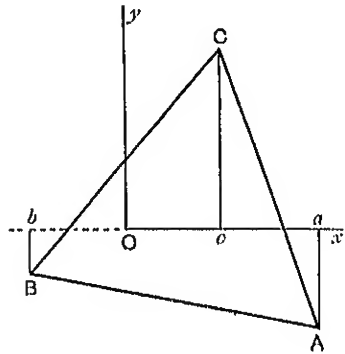
COR. I. If Ox and Oy are two rectangular axes and if a point moves from O to P, a distance r units, in the direction θ, then it could have arrived at the same point by firstly moving from O a distance $r\cos\theta$ in the direction Ox and secondly moving a distance $r\sin\theta$ in the direction Oy.

COR. II. If Ox and Oy are two rectangular axes and a point, starting from *any point*, moves a distance r units in the direction θ, its coordinates are *algebraically increased* by

$$r\cos\theta \text{ and } r\sin\theta.$$

Theorem II. *If* Ox *and* Oy *are two rectangular axes, and* A, B, C *any three points, then if a point moves from* A *to* B, *and then from* B *to* C, *the total increase in its x-coordinate is the same as it would have been, had the point moved directly from* A *to* C.





Let Aa, Bb, Cc be the perpendiculars from A, B, C on Ox.

Then, as the point moves from A to B the increase of its abscissa is

$$Ob - Oa,$$

and as it moves from B to C, the increase of its abscissa is

$$Oc - Ob.$$

When the point moves from A to C directly, the increase of its abscissa is

$$Oc - Oa.$$

$\therefore$ total increase in abscissa in moving from A to C through B

$$= (Ob - Oa) + (Oc - Ob)$$

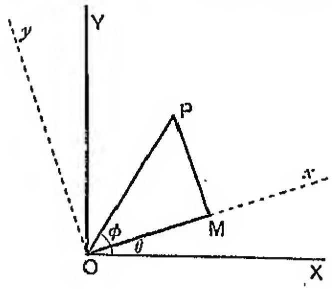
$$= Oc - Oa$$

$=$ increase in moving from A to C directly.

Theorem III. *For all magnitudes of θ and ϕ, positive or negative,*

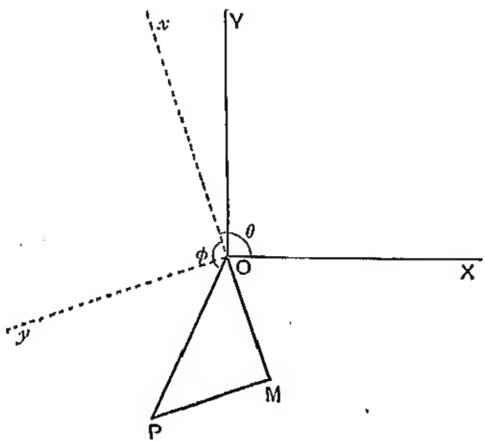
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Let OX and OY be two rectangular axes and suppose these axes to rotate about O through an angle θ so that they take up the new positions Ox and Oy.



Let OP be drawn in the direction ϕ with respect to the axes Ox and Oy, and let OP contain r units of length, r being positive; then with respect to OX and OY, OP is in the direction $(\theta + \phi)$.

Draw PM perpendicular to Ox.



Then (i) a point could travel from O to P by passing through M, and would move a distance

$r \cos \phi$ in the direction Ox,

and $r \sin \phi$ in the direction Oy. (Th. I., Cor. I.)

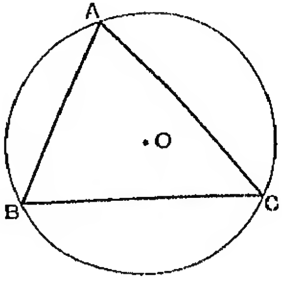
Hence the point could travel from O to P by moving, with respect to OX and OY, a distance

$r \cos \phi$ in the direction θ,

$r \sin \phi$ in the direction $(\theta + 90°)$.

142. *To find the radius of the circumcircle of a triangle.*

We have already proved in Art. 73,



that $$\mathrm{R}=\frac{a}{2\sin \mathrm{A}}.$$

Now $$\frac{a}{2\sin \mathrm{A}}=\frac{abc}{4\times\frac{1}{2}bc\sin \mathrm{A}}$$
$$=\frac{abc}{4\Delta};$$
$$\therefore\ \mathrm{R}=\frac{abc}{4\Delta}.$$

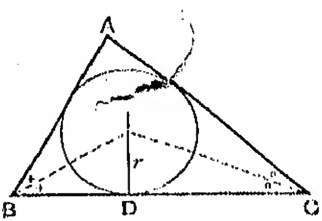
143. *To find the radius of the in-circle of a triangle.*

The in-centre I being found by bisecting two angles of the triangle by the lines BI, CI, a perpendicular ID is drawn to the side BC.

We have already proved in Arts. 79 and 80 that
$$r=\frac{\Delta}{s}=(s-a)\tan\frac{\mathrm{A}}{2}.$$

Also

$$\frac{r}{a}=\frac{r}{\mathrm{IB}}\cdot\frac{\mathrm{IB}}{a}=\sin\frac{\mathrm{B}}{2}\cdot\frac{\sin\frac{\mathrm{C}}{2}}{\sin \mathrm{BIC}}$$
$$=\frac{\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}{\sin\left(\pi-\frac{\mathrm{B}+\mathrm{C}}{2}\right)}$$
$$=\frac{\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}}{\cos\frac{\mathrm{A}}{2}},$$



$$\therefore\ r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = \frac{2R \sin A \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$$

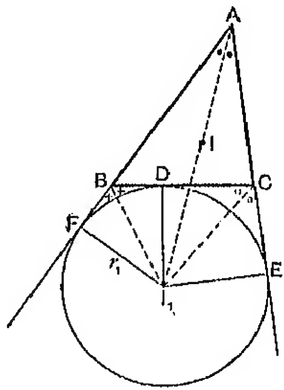
$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

Note that
$$\frac{IB}{a} = \frac{\sin \frac{C}{2}}{\cos \frac{A}{2}},$$

$$\therefore\ IB = 4R \sin \frac{A}{2} \sin \frac{C}{2}, \text{ etc.}$$

144. *To find the radius of an escribed circle of a triangle.*

The *e*-centre opposite the angle A is found by bisecting the exterior angles CBF, BCE by the lines BI_1 and CI_1. Perpendiculars I_1D, I_1E, I_1F are then drawn to the sides of the triangle.



$$\Delta = ABI_1C - \text{area of } BI_1C$$
$$= ABI_1 + ACI_1 - BI_1C$$
$$= \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1$$
$$= r_1\left(\frac{b+c-a}{2}\right)$$
$$= r_1\left(\frac{b+c+a}{2} - a\right)$$
$$= r_1(s-a);$$

$$\therefore\ r_1 = \frac{\Delta}{s-a}.$$

Similarly if r_2 and r_3 are the radii of the *e*-circles opposite the angles B and C respectively

$$r_2 = \frac{\Delta}{s-b}, \qquad r_3 = \frac{\Delta}{s-c}.$$

145. Let $BD (= BF) = x$; $CD (= CE) = a - x$.

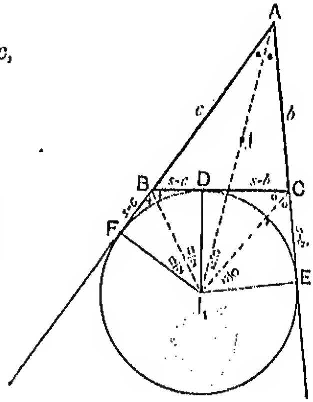
$\therefore AF = c + x$ and $AE = b + a - x$,

$$\therefore c + x = b + a - x,$$

$$\therefore BD (= BF) = x = \frac{b + a - c}{2} = s - c,$$

$$\therefore AF (= AE) = c + x = s,$$

$$CD (= CE) = a - x = s - b.$$



Since the angles BFI_1 and BDI_1 are right angles, it follows that the points I_1FBD are concyclic.

$$\therefore A\hat{B}C (= B) = F\hat{I_1}D$$

$$\therefore F\hat{I_1}B = \frac{B}{2}.$$

Similarly $$E\hat{I_1}C = \frac{C}{2}.$$

If I is the in-centre, then AII_1 is a straight line.

$$r_1 = AF \tan\frac{A}{2}$$

$$= s \tan\frac{A}{2}.$$

Also $$r_1 = BF \cot FI_1B = (s - c)\cot\frac{B}{2}$$

$$= EC \cot EI_1C = (s - b)\cot\frac{C}{2}.$$

Similarly $$r_2 = s\tan\frac{B}{2} = (s - a)\cot\frac{C}{2} = (s - c)\cot\frac{A}{2}$$

$$r_3 = s\tan\frac{C}{2} = (s - a)\cot\frac{B}{2} = (s - b)\cot\frac{A}{2}.$$

146. $\dfrac{r_1}{a} = \dfrac{r_1}{I_1B} \cdot \dfrac{I_1B}{a} = \sin FBI_1 \cdot \dfrac{\sin BCI_1}{\sin BI_1C}$

$$= \cos\frac{B}{2} \cdot \frac{\cos\frac{C}{2}}{\sin\frac{B+C}{2}};$$

$$\therefore\ r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{b\sin A}{\sin B} \cdot \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} \text{ or } \frac{c\sin A}{\sin C} \cdot \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{b\sin\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{B}{2}} \text{ or } \frac{c\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}.$$

Also
$$r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{2R\sin A\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

with corresponding expressions for r_2 and r_3.

Note that
$$\frac{I_1B}{a} = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

$$\therefore\ I_1B = 4R\sin\frac{A}{2}\cos\frac{C}{2}, \text{ etc.}$$

147. Ex. 1. Prove that $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}$.

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$$

$$=\frac{3s-(a+b+c)}{\Delta}$$

$$=\frac{s}{\Delta}$$

$$=\frac{1}{r}.$$

Ex. 2. Show that $\dfrac{b^2-c^2}{2}\cdot\dfrac{\sin B\sin C}{\sin(B-C)}=\Delta$.

$$\text{Expression}=\frac{4R^2(\sin^2 B-\sin^2 C)}{2}\cdot\frac{\sin B\sin C}{\sin(B-C)}$$

$$=\frac{4R^2\sin(B+C)}{2}\cdot\sin B\sin C$$

$$=\frac{4R^2\sin A\sin B\sin C}{2}$$

$$=\tfrac{1}{2}bc\sin A$$

$$=\Delta.$$

EXAMPLES XXXV.

Prove that

1. $\Delta=\sqrt{rr_1r_2r_3}$.

2. $\Delta=\dfrac{arr_1}{r_1-r}$.

3. $\Delta=\dfrac{(b+c)rr_1}{r+r_1}$.

4. $\Delta=\dfrac{ar_2r_3}{r_2+r_3}$.

5. $\Delta=\dfrac{rr_1(r_2-r_3)}{b-c}$.

6. $\Delta=\dfrac{rr_2\sqrt{r_1+r_3}}{\sqrt{r_2-r}}$.

7. $\Delta=r_2r_3\tan\dfrac{A}{2}$.

8. $\Delta=rr_1\cot\dfrac{A}{2}$.

9. $\Delta = r_1 r_2 r_3 / \sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}$.

10. $\Delta = r\sqrt{r_1 r_2 + r_2 r_3 + r_3 r_1}$.

11. $\Delta = \dfrac{r}{2R^{\frac{1}{2}}}\sqrt{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}$.

12. $rs^2 = r_1 r_2 r_3$.

13. $r_1 r_2 r_3 = r^3 \cot^2 \dfrac{A}{2} \cot^2 \dfrac{B}{2} \cot^2 \dfrac{C}{2}$.

14. $4Rrs = abc$.

15. $4R = r_1 + r_2 + r_3 - r$.

16. $r_3 = r \cot \dfrac{A}{2} \cot \dfrac{B}{2}$.

17. $rr_1 = r_2 r_3 \tan^2 \dfrac{A}{2}$.

18. $a(rr_1 + r_2 r_3) = b(rr_2 + r_3 r_1) = c(rr_3 + r_1 r_2)$.

19. $\left(\dfrac{r_1}{r} - 1\right)\left(\dfrac{r_2}{r} - 1\right)\left(\dfrac{r_3}{r} - 1\right) = \dfrac{4R}{r}$.

20. $2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)$.

21. $2(R + r) = a \cot A + b \cot B + c \cot C$.

22. $\Delta^2\left(\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2}\right) = a^2 + b^2 + c^2$.

23. $4Rr + r^2 = ab + bc + ca - s^2$.

24. $\dfrac{1}{c \sin B} + \dfrac{1}{a \sin C} + \dfrac{1}{b \sin A} = \dfrac{1}{r}$.

25. $4\Delta(\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$.

26. $(b - c) r_2 r_3 + (c - a) r_3 r_1 + (a - b) r_1 r_2 = 0$.

27. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a \sin A + b \sin B + c \sin C}{4\Delta}$.

28. $2R(1 - \cos A) = r_1 - r$.

29. $\dfrac{\cos A}{c \sin B} + \dfrac{\cos B}{a \sin C} + \dfrac{\cos C}{b \sin A} = \dfrac{1}{R}$.

30. Find the radius of the inscribed circle of a triangle whose sides are 706, 690 and 240 feet.

31. If the sides of a triangle are 3, 4, 5 inches in length, in what ratio do the points of contact of the inscribed circle divide the sides?

32. If the sides of a triangle are 5, 6 and 9 centimetres in length, find the radius of the circum-circle.

Prove that

33. $r_1(\cos B - \cos C) + r_2(\cos C - \cos A) + r_3(\cos A - \cos B) = 0.$

34. $\dfrac{r_3^2}{4R - r_1 - r_2} = r_3 + \dfrac{r_1 r_2}{r_1 + r_2}.$

35. $p_1 \cos A + p_2 \cos B + p_3 \cos C = 2R(1 + \cos A \cos B \cos C),$

where p_1, p_2, p_3 are the perpendiculars from A, B, C on the opposite sides of the triangle ABC.

36. $abc + (a-b)(b-c)(c-a) = 4Rr(a\cos C + b\cos A + c\cos B).$

37. $8R^2(1 + \cos A \cos B \cos C) = a^2 + b^2 + c^2.$

38. $a^2r^2 - 2a^2\Delta r + a(r^4 + 4r^3R + \Delta^2) - 4\Delta Rr^2 = 0.$

39. If p is the perpendicular from the angle A on to BC,

$$p = r \operatorname{cosec} \frac{A}{2} \sqrt{(1 + \cos B)(1 + \cos C)}.$$

40. If in the ambiguous case of the solution of a triangle where a, c and C are given, the two values of b are b_1 and b_2 and r_1, r_2 be the radii of the corresponding in-circles, prove that

$$\left(\frac{b_1}{r_1} - \cot\frac{C}{2}\right)\left(\frac{b_2}{r_2} - \cot\frac{C}{2}\right) = 1,$$

and
$$r_1 r_2 = a(a - c)\sin^2\frac{C}{2}.$$

41. $$\Delta = r_1 r_2 \sqrt{\frac{4R - r_1 - r_2}{r_1 + r_2}}.$$

42. Prove that if θ is the angle at which the perpendicular from the vertex A to the side BC of a triangle ABC cuts the inscribed circle, then

$$\cos\theta = \sin\tfrac{1}{2}(B - C)\operatorname{cosec}\tfrac{1}{2}A.$$

43. Prove that

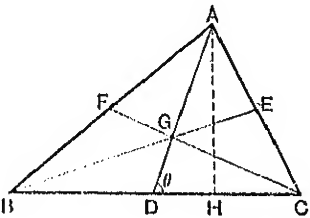
$$\sin^2 A + \sin B \sin C \cos A = 2\Delta^2(a^2+b^2+c^2)/a^2b^2c^2.$$

44. Prove that if the bisector of the angle C cuts AB in D and the circum-circle in E,

$$CE/DE = (a+b)^2/c^2.$$

148. Medians.

If AD, BE and CF are the medians, then G the point of intersection is known as the *Centroid*, and by Elementary Geometry $\frac{AG}{GD} = \frac{2}{1}$, etc.



Also

$$2AD^2 + 2BD^2 = AB^2 + AC^2,$$

$$\therefore \quad 2AD^2 = c^2 + b^2 - \frac{a^2}{2}$$

or

$$AD^2 = \frac{1}{2}\left(b^2 + c^2 - \frac{a^2}{2}\right).$$

Similarly

$$BE^2 = \frac{1}{2}\left(c^2 + a^2 - \frac{b^2}{2}\right)$$

$$CF^2 = \frac{1}{2}\left(a^2 + b^2 - \frac{c^2}{2}\right).$$

149. If θ is the angle that AD makes with BC, and AH is perpendicular to BC,

$$DH = \tfrac{1}{2}[(BD + DH) - (DC - DH)]$$
$$= \tfrac{1}{2}(BH - HC);$$

$$\therefore \quad \cot\theta = \frac{DH}{AH} = \frac{1}{2}\,\frac{BH - HC}{AH}$$
$$= \tfrac{1}{2}(\cot B - \cot C).$$

Also

$$\cot\theta = \frac{DH}{AH} = \frac{DC - HC}{AH}$$
$$= \frac{\frac{a}{2} - b\cos C}{b\sin C}$$

$$=\frac{a^2-2ab\cos C}{2ab\sin C}$$

$$=\frac{a^2-(a^2+b^2-c^2)}{2ab\sin C}$$

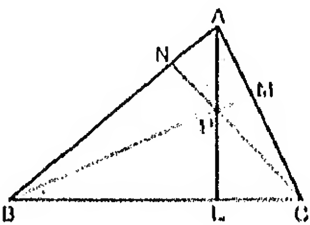
$$=\frac{c^2-b^2}{4\Delta}.$$

The Pedal Triangle.

150. The pedal triangle LMN is obtained by joining the feet of the perpendiculars from the angular points of a triangle to the opposite sides.

The point of intersection, P, of these perpendiculars is called the *Orthocentre.*

151. *To find the distances of the orthocentre from the angles and sides of the triangle.*



$$\mathrm{PL}=\mathrm{BL}\tan \mathrm{PBL}=\mathrm{AB}\cos \mathrm{B}\cot \mathrm{MCB}$$

$$=c\cos \mathrm{B}\cot \mathrm{C}$$

$$=\frac{c}{\sin \mathrm{C}}\cos \mathrm{B}\cos \mathrm{C}$$

$$=2\mathrm{R}\cos \mathrm{B}\cos \mathrm{C}.$$

Similarly,

$$\mathrm{PM}=2\mathrm{R}\cos \mathrm{C}\cos \mathrm{A},$$

$$\mathrm{PN}=2\mathrm{R}\cos \mathrm{A}\cos \mathrm{B}.$$

$$\mathrm{PA}=\mathrm{AM}\sec \mathrm{PAM}$$

$$=\mathrm{AB}\cos \mathrm{A}\operatorname{cosec} \mathrm{ACL}$$

$$=\frac{c}{\sin \mathrm{C}}\cos \mathrm{A}$$

$$=2\mathrm{R}\cos \mathrm{A}.$$

Similarly,

$$\mathrm{PB}=2\mathrm{R}\cos \mathrm{B},$$

$$\mathrm{PC}=2\mathrm{R}\cos \mathrm{C}.$$

152. *To find the angles and sides of the Pedal Triangle.*

Since BNMC is concyclic

$\therefore\ A\hat{N}M = 180° - BNM = C,$

$A\hat{M}N = 180° - NMC = B.$

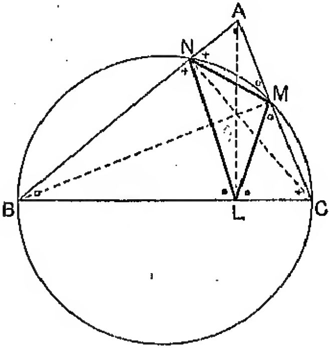
Similarly

$B\hat{N}L = C$ etc.

$\therefore\ M\hat{N}L = 180° - 2C,$

$N\hat{L}M = 180° - 2A,$

$L\hat{M}N = 180° - 2B.$



Since BC ($= a$) is the diameter of the circle through BNMC

$\therefore\ MN = a \sin NBM$ (Art. 73)

$= a \cos A = 2R \sin A \cos A = R \sin 2A.$

Similarly $NL = b \cos B = R \sin 2B,$

$LM = c \cos C = R \sin 2C.$

153. *To find the area of the Pedal Triangle and the radius of its circum-circle.*

Area of $LMN = \frac{1}{2} NL \cdot NM \sin LNM$

$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C.$

Radius of circum-circle

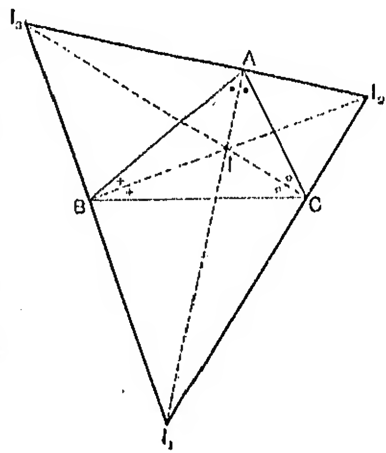
$$= \frac{\text{any side}}{2 \sin (\text{opposite angle})} = \frac{MN}{2 \sin MLN}$$

$$= \frac{R \sin 2A}{2 \sin 2A} = \frac{R}{2}.$$

T.A. Natarajan

The Ex-central Triangle.

154. If I_1, I_2, I_3 are the *e*-centres of the triangle ABC, then $I_1I_2I_3$ is called the Ex-central Triangle of ABC. By Geometry, AII_1, BII_2, CII_3, are straight lines as are also I_2AI_3, I_3BI_1, I_1CI_2, the first three being respectively perpendicular to the second three.



Thus ABC is the pedal triangle of $I_1I_2I_3$.

By making use of the results obtained for the Pedal Triangle we can thus obtain the properties of the Ex-central Triangle.

$$B\hat{A}C = 180^\circ - 2I_3\hat{I}_1I_2$$

$$\therefore \; I_3\hat{I}_1I_2 = 90^\circ - \frac{A}{2}.$$

Similarly

$$I_1\hat{I}_2I_3 = 90^\circ - \frac{B}{2}$$

$$I_2\hat{I}_3I_1 = 90^\circ - \frac{C}{2},$$

$$BC = I_2I_3 \cos I_3I_1I_2$$

$$= I_2I_3 \cos\left(90^\circ - \frac{A}{2}\right),$$

$$\therefore \; I_2I_3 = \frac{a}{\sin\dfrac{A}{2}}.$$

Similarly
$$I_3I_1 = \frac{b}{\sin\frac{B}{2}}$$

$$I_1I_2 = \frac{c}{\sin\frac{C}{2}}.$$

The values may easily be proved equal to

$$4R\cos\frac{A}{2},\quad 4R\cos\frac{B}{2},\quad 4R\cos\frac{C}{2}.$$

155. Area of $I_1I_2I_3 = \frac{1}{2}\,I_1I_3\,.\,I_1I_2 \sin I_3I_1I_2$

$$= \tfrac{1}{2}\,.\,16R^2\cos\frac{B}{2}\cos\frac{C}{2}\sin\left(90^\circ - \frac{A}{2}\right)$$

$$= 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Radius of circum-circle of ABC $= \frac{1}{2}$ radius of circum-circle of $I_1I_2I_3$.

$\therefore$ Rad. of circum-circle of $I_1I_2I_3 = 2R$.

The Bisectors of the Angles.

(iv) **156.** Let AK and AK′ be the bisectors of internal angle BAC and its supplement respectively.

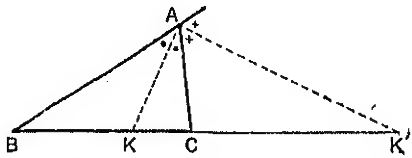
$$\frac{BK}{KC} = \frac{BA}{AC} = \frac{c}{b},$$

$$\therefore \frac{BK}{BK+KC} = \frac{c}{b+c},$$

$$\therefore BK = \frac{ac}{b+c}.$$

Similarly $$KC = \frac{ab}{b+c}$$

$$\frac{BK'}{K'C} = \frac{BA}{AC} = \frac{c}{b},$$



$$\therefore \frac{BK'}{BK' - K'C} = \frac{c}{c-b},$$

$$\therefore BK' = \frac{ac}{c-b}.$$

Similarly $$CK' = \frac{ab}{c-b}.$$

157. *To find the lengths of the bisectors.*

$$\triangle ABK + \triangle AKC = \triangle ABC.$$

$$\therefore \tfrac{1}{2}AB \,.\, AK \sin\frac{A}{2} + \tfrac{1}{2}AK \,.\, AC \sin\frac{A}{2} = \tfrac{1}{2}AB \,.\, AC \sin A$$

$$AK\,(c+b)\sin\frac{A}{2} = bc\sin A$$

$$AK = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

Similarly $$\triangle ABK' - \triangle ACK' = \triangle ABC.$$

$$\tfrac{1}{2}AB \,.\, AK' \sin BAK' - \tfrac{1}{2}AC \,.\, AK' \sin CAK' = \tfrac{1}{2}AB \,.\, AC \sin A$$

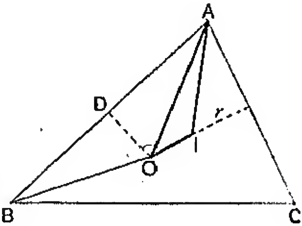
$$AK'(c-b)\cos\frac{A}{2} = bc\sin A,$$

$$\therefore AK' = \frac{2bc}{c-b}\sin\frac{A}{2}.$$

[These results have previously been proved in Art. 84.]

158. *To find the distance between the in-centre and the circum-centre.*

If I is the in-centre and O the circum-centre



$$I\hat{A}D = \frac{A}{2}$$

$$O\hat{A}D = 90^\circ - AOD = 90^\circ - C;$$

$$\therefore\ I\hat{A}O = \frac{A}{2} - 90^\circ + C$$

$$= \frac{A-(A+B+C)+2C}{2}$$

$$= \frac{C-B}{2}$$

$$AO = R$$

$$AI = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2} \text{ (Art. 143).}$$

Therefore from the triangle IOA,

$$OI^2 = AO^2 + AI^2 - 2AO\,.\,AI\cos OAI$$

$$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$$= R^2 + 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\left[2\sin\frac{B}{2}\sin\frac{C}{2} - \cos\frac{C}{2}\cos\frac{B}{2} - \sin\frac{C}{2}\sin\frac{B}{2}\right]$$

$$= R^2 - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= R^2\left[1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}\right]$$

$$= R^2 - 2R\,.\,4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= R^2 - 2Rr.$$

159. Similarly if I_1, I_2, I_3 are the *e*-centres, we have

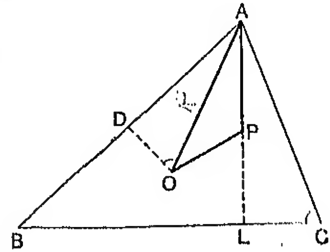
$$OI_1^2 = R^2 + 2Rr_1$$

$$OI_2^2 = R^2 + 2Rr_2$$

$$OI_3^2 = R^2 + 2Rr_3.$$

160. *To find the distance between the circum-centre and the orthocentre.*

Let O be the circum-centre and P the orthocentre.



$$O\hat{A}D = 90° - AOD = 90° - C$$

$$P\hat{A}D = 90° - ABL = 90° - B,$$

$$\therefore\ P\hat{A}O = P\hat{A}D - O\hat{A}D$$
$$= C - B$$

$$AO = R$$

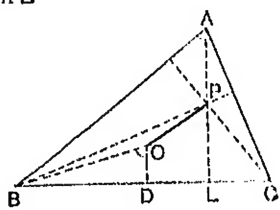
$AP = 2R\cos A$ (Art. 151).

From the triangle OAP,

$$OP^2 = AO^2 + AP^2 - 2AO\,.\,AP\cos OAP$$
$$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$$
$$= R^2 + 4R^2\cos A\,[\cos A - \cos(C-B)]$$
$$= R^2 + 4R^2\cos A\,[-\cos(C+B) - \cos(C-B)]$$
$$= R^2 - 8R^2\cos A\cos B\cos C$$
$$= R^2[1 - 8\cos A\cos B\cos C].$$

161. Ex. 1. Prove that the line joining O and P makes with BC an angle θ, where

$$\tan\theta = \frac{3 - \tan B\tan C}{\tan C - \tan B}.$$



$PL = a\cos B\cot C$ (Art. 151),

$$OD = BD\cot A = \frac{a}{2}\cot A,$$

$$DL = \tfrac{1}{2}[(BD + DL) - (CD - DL)]$$
$$= \tfrac{1}{2}(BL - CL)$$
$$= \tfrac{1}{2}(a\cos B - b\cos C),$$

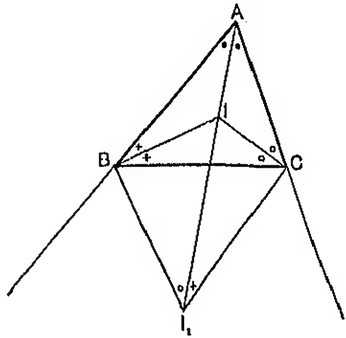
$$\tan\theta = \frac{PL - OD}{DL} = \frac{2a\cos B\cot C - a\cot A}{a\cos B - b\cos C}$$

$$= \frac{2\cos B\cos C - \cos A}{\sin C\cos B - \sin B\cos C}$$

$$= \frac{3\cos B\cos C - \sin B\sin C}{\sin C\cos B - \sin B\cos C} \qquad [\text{since } \cos A = -\cos(B+C)]$$

$$= \frac{3 - \tan B\tan C}{\tan C - \tan B}.$$

Ex. 2. Prove that $\dfrac{II_1^2}{r_1 - r} = 4R.$



$B\hat{I_1}C = 90^\circ - \dfrac{A}{2}$. (Art. 154.)

$I\hat{B}I_1 = I\hat{C}I_1 = 90^\circ$.

∴ IBI_1C is concyclic, II_1 being a diameter.

∴ $BC = II_1 \sin BI_1C$ (Art. 73)

$$= II_1 \cos\frac{A}{2}.$$

$$\therefore\ II_1 = \frac{a}{\cos\frac{A}{2}}.$$

Also
$$\sin\frac{A}{2} = \frac{r_1}{AI_1} = \frac{r}{AI} = \frac{r_1 - r}{II_1},$$

$$4R = \frac{2a}{\sin A} = \frac{a}{\cos\frac{A}{2}}\cdot\frac{1}{\sin\frac{A}{2}} = II_1\cdot\frac{II_1}{r_1 - r}$$

$$= \frac{II_1^2}{r_1 - r}.$$

EXAMPLES XXXVI.

If I, I_1, I_2, I_3, O, P be the in-centre, e-centres, circum-centre, and orthocentre of a triangle ABC,

Prove that

1. $\dfrac{IA\,.\,IB}{IC} = 4R\sin^2\dfrac{C}{2}.$

2. $\dfrac{I_1A\,.\,I_1B}{I_1C} = 4R\cos^2\dfrac{C}{2}.$

3. $IA\,.\,IB\,.\,IC = 4\Delta R\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}.$

4. $\dfrac{II_1}{I_2I_3} = \tan\dfrac{A}{2}.$

5. Area of $I_1 I_2 I_3 = \dfrac{abc}{2r}$.

6. $\dfrac{\text{Area of } I_2 I_3 I_1}{\text{Area of } I_3 I_1 I} = \dfrac{r_2}{r}$.

7. If β and γ are the angles the median through A makes with AB and AC, then $c \sin \beta = b \sin \gamma$.

8. $IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

9. The radius of the inscribed circle of the pedal triangle is

$$2R \cos A \cos B \cos C.$$

10. $IA^2 + I_1A^2 + I_2A^2 + I_3A^2 = 16R^2$.

11. $IO^2 + I_1O^2 + I_2O^2 + I_3O^2 = 12R^2$.

12. $a \,.\, BP \,.\, CP + b \,.\, CP \,.\, AP + c \,.\, AP \,.\, BP = abc$.

13. Area of triangle

$IOP = -2R^2 \sin \frac{1}{2}(B - C) \sin \frac{1}{2}(C - A) \sin \frac{1}{2}(A - B)$.

14. $\Delta = r^2 \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C$.

15. $IA \,.\, IB \,.\, IC = 4Rr^2$.

16. If x, y, z are the perpendiculars from O to the sides of the triangle

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

17. $\dfrac{IA}{I_1A} + \dfrac{IB}{I_2B} + \dfrac{IC}{I_3C} = 1$.

18. If the perpendiculars AL, BM, CN from the angular points to the opposite sides, meet the circum-circle again in L′, M′, N′,

$$\frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4.$$

19. If the line IO makes an angle θ with BC,

$$\cot \theta = \frac{\sin B - \sin C}{\cos B + \cos C - 1}.$$

20. If the bisectors of the angles B and C meet the opposite sides in E, F, and the line EF makes an angle ϕ with BC,

$$\tan\phi = \frac{(b \sim c)\sin A}{(a+b)\cos C + (a+c)\cos B} = \frac{\sin B \sim \sin C}{\cos B + \cos C + 1}.$$

21. $$\frac{r_1 r_2 r_3}{r^3} = \frac{(a+b+c)^3}{(b+c-a)(c+a-b)(a+b-c)}.$$

22. If AL, BM, CN are the perpendiculars from the angular points to the opposite sides,

(i) the perimeter of LMN $= 4R \sin A \sin B \sin C$,

(ii) $$\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

23. Two circles are described with centres at the corners A, B of an acute-angled triangle ABC, so as to touch the sides BC, CA respectively. Prove that the angle θ at which the circles cut is given by

$$\cos\theta = \tfrac{1}{2}\cos C(\cot A \cot B + 1).$$

24. Prove that the diameter of the circum-circle through A is divided by BC in the ratio of $\tan B \tan C : 1$.

25. Perpendiculars from A, B, C on the opposite sides meet the circum-circle again in D, E, F. Prove that the ratio of the area of triangle DEF to that of ABC is $8\cos A \cos B \cos C$.

26. The inscribed circle touches BC at D and the perpendicular from A on BC meets BC in E. Prove that

$$DE = \frac{(b \sim c)(b+c-a)}{2a}.$$

27. If AD is drawn perpendicular to BC and if ρ_1, ρ_2 denote the radii of the inscribed circles of the triangles ABD, ACD, show that

$$\frac{\cot B}{\rho_1} + \frac{\cot C}{\rho_2} = (\cot B + \cot C)\left(\frac{1}{r} + \frac{2}{a}\right).$$

28. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides, and Δ_1, Δ_2, Δ_3 corresponding areas for the escribed circles

$$s\Delta_0 = (s-a)\Delta_1 = (s-b)\Delta_2 = (s-c)\Delta_3.$$

29. If G is the intersection of the medians of a triangle ABC (area Δ), prove that

$$9AG^2 = 2\Delta(4\cot A + \cot B + \cot C).$$

30. Prove that

$$OP^2 = 2R^2(\tfrac{3}{2} + \cos 2A + \cos 2B + \cos 2C).$$

31. If K is the centre of the circle circumscribing IBC, prove that

$$IK^2 = (R+r)^2 + r^2 - \frac{2\Delta}{r_2 r_3}.$$

32. If D, E, F are the mid-points of the sides of a triangle ABC, and D′, E′, F′ the feet of the perpendiculars from the vertices A, B, C on the opposite sides, prove that

$$\frac{a^2 \cos B \cos C}{EE' \cdot FF'} + \frac{b^2 \cos C \cos A}{FF' \cdot DD'} + \frac{c^2 \cos A \cos B}{DD' \cdot EE'} = 4.$$

33. Prove that $(a+b+c)\,H_1 \cdot H_2 \cdot H_3 = [illegible]$.

34. Given an isosceles triangle whose vertical angle is A, and base a, show that the diameter of the circle which cuts the sides of the triangle two and two in points which are at the opposite extremities of a diameter is

$$\frac{a}{2 - \cos A}.$$

35. If U is the centre of the nine-point circle of a triangle ABC, prove that

$$IU = \tfrac{1}{2}R - r.$$

36. On the base BC of a triangle ABC, a point V is taken such that VC/VB = sin 2C/sin 2B, whilst the line joining the circum-centre O and the orthocentre P meets BC at T. If VK be the perpendicular from V on OP, and if OP be bisected at I, then

$$4IK \cdot IT = R^2.$$

37. Show that the orthocentre of a triangle lies on the inscribed circle if

$$\cos A \cos B \cos C = 4 \sin^2\frac{A}{2} \sin^2\frac{B}{2} \sin^2\frac{C}{2}.$$

38. If the line joining the circum-centre and in-centre of a triangle touches the escribed circle opposite the angle A, prove that

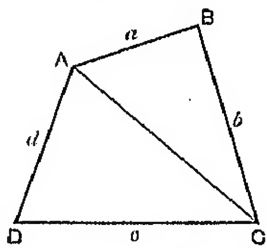
$$\Delta(r_2 - r_3) = r_1 r_2 r_3\left(1 - \frac{2r}{R}\right)^{\frac{1}{2}}.$$

CHAPTER XV.

QUADRILATERALS AND POLYGONS.

162. *To find the area* (S) *of a quadrilateral.*

Let $\quad B + D = 2\alpha \quad$ and $\quad a + b + c + d = 2s.$



$$AC^2 = a^2 + b^2 - 2ab\cos B$$
$$= c^2 + d^2 - 2cd\cos D,$$

$$\therefore\ a^2 + b^2 - c^2 - d^2$$
$$= 2(ab\cos B - cd\cos D) \ldots\ldots \text{(i)}.$$

Also

$$4S = 4ABC + 4ACD$$
$$= 2(ab\sin B + cd\sin D) \ldots\ldots \text{(ii)}.$$

$\therefore$ squaring and adding,

$$16S^2 + (a^2 + b^2 - c^2 - d^2)^2$$
$$= 4\left[a^2b^2 + c^2d^2 - 2abcd\cos(B + D)\right]$$
$$= 4\left[a^2b^2 + c^2d^2 - 2abcd\cos 2\alpha\right]$$
$$= 4\left[a^2b^2 + c^2d^2 - 2abcd(2\cos^2\alpha - 1)\right]$$
$$= 4(ab + cd)^2 - 16abcd\cos^2\alpha;$$

$$\therefore\quad 16S^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd\cos^2\alpha$$
$$= \left[2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\right]$$
$$\left[2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\right] - 16abcd\cos^2\alpha$$
$$= \left[(a + b)^2 - (c - d)^2\right]\left[(c + d)^2 - (a - b)^2\right] - 16abcd\cos^2\alpha$$
$$= (a + b + c - d)(a + b - c + d)(c + d + a - b)$$
$$(c + d - a + b) - 16abcd\cos^2\alpha$$
$$= (2s - 2d)(2s - 2c)(2s - 2b)(2s - 2a) - 16abcd\cos^2\alpha;$$

$$\therefore\quad \mathbf{S^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha.}$$

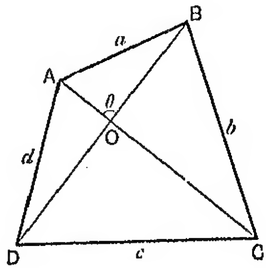
In the case of a cyclic quadrilateral,

$$B + D = 2\alpha = 180°,$$

$$\therefore \cos\alpha = 0.$$

Thus $$S^2 = (s-a)(s-b)(s-c)(s-d).$$

163. The area may also be found in terms of the diagonals and the angle between them.



$$2S = 2\triangle AOB + 2\triangle BOC + 2\triangle AOD + 2\triangle DOC$$
$$= AO \,.\, OB\sin\theta + BO \,.\, OC\sin(\pi-\theta)$$
$$+ AO \,.\, OD\sin(\pi-\theta) + DO \,.\, OC\sin\theta$$
$$= AO \,.\, DB\sin\theta + BD \,.\, OC\sin\theta,$$

$$\therefore S = \tfrac{1}{2}AC \,.\, DB\sin\theta.$$

164. In the case of a cyclic quadrilateral, since B and D are supplementary, equation (i), Art. 162, becomes

$$a^2 + b^2 - c^2 - d^2 = 2(ab+cd)\cos B,$$

i.e. $$\cos B = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$$

and from (ii) $$\sin B = \frac{2S}{ab+cd},$$

165. *To find the diagonals and circum-radius* (R) *of a cyclic quadrilateral.*

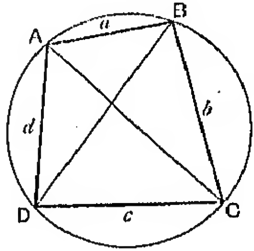
We have shown that $AC^2 = a^2 + b^2 - 2ab\cos B$.

Substituting for cos B from Art. 164 we have

$$AC^2 = a^2 + b^2 - \frac{ab(a^2+b^2-c^2-d^2)}{ab+cd}$$

$$= \frac{cd(a^2+b^2)+ab(c^2+d^2)}{ab+cd}$$

$$= \frac{(ac+bd)(ad+bc)}{ab+cd}.$$



Similarly

$$BD^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}.$$

The circle circumscribing ABCD also circumscribes the triangle ABC;

$$\therefore\ R = \frac{AC}{2\sin B} = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}} \cdot \frac{ab+cd}{4S}\quad \text{(Art. 164)}$$

$$= \frac{1}{4S}\sqrt{(ab+cd)(ac+bd)(ad+bc)}.$$

166. Ex. 1. Find the area of a cyclic quadrilateral when the sides are 4, 5, 7, 8 centimetres respectively.

$$s = \frac{4+5+7+8}{2} = 12.$$

$$\therefore\ S = \sqrt{8 \cdot 7 \cdot 5 \cdot 4}\ \text{sq. cms.}$$

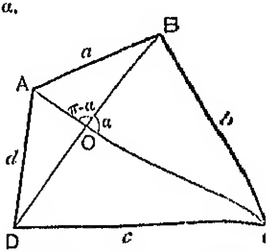
$$= 4\sqrt{70}\ \text{sq. cms.}$$

$$= 33{\cdot}46\ \text{sq. cms.}$$

(correct to the nearest sq. millimetre).

Ex. 2. If a, b, c, d are the sides of a quadrilateral and α the angle opposite b between the diagonals, prove that the area o the quadrilateral is

$$\tfrac{1}{4}(a^2+c^2-b^2-d^2)\tan\alpha.$$



$$2\mathrm{OC}\,.\,\mathrm{OB}\cos\alpha = \mathrm{OC}^2+\mathrm{OB}^2-b^2$$

$$2\mathrm{OA}\,.\,\mathrm{OB}\cos(\pi-\alpha) = \mathrm{OA}^2+\mathrm{OB}^2-a^2.$$

$\therefore$ subtracting

$$2\mathrm{AC}\,.\,\mathrm{OB}\cos\alpha = \mathrm{OC}^2-\mathrm{OA}^2-b^2+a^2 \;\ldots\ldots\text{(i)}.$$

Also

$$2\mathrm{OA}\,.\,\mathrm{OD}\cos\alpha = \mathrm{OA}^2+\mathrm{OD}^2-d^2$$

$$2\mathrm{OC}\,.\,\mathrm{OD}\cos(\pi-\alpha) = \mathrm{OC}^2+\mathrm{OD}^2-c^2.$$

Subtracting, $\quad 2\mathrm{AC}\,.\,\mathrm{OD}\cos\alpha = \mathrm{OA}^2-\mathrm{OC}^2-d^2+c^2 \;\ldots\ldots\ldots\ldots\text{(ii)}.$

$\therefore$ adding (i) and (ii),

$$2\mathrm{AC}\,.\,\mathrm{BD}\cos\alpha = a^2+c^2-b^2-d^2.$$

Now $\quad 2\mathrm{AC}\,.\,\mathrm{BD}\sin\alpha = 4\mathrm{S}, \qquad$ (Art. 163)

$$\therefore\ \tan\alpha = \frac{4\mathrm{S}}{a^2+c^2-b^2-d^2},$$

or $\quad \mathrm{S} = \tfrac{1}{4}(a^2+c^2-b^2-d^2)\tan\alpha.$

EXAMPLES XXXVII.

1. If the sides of a cyclic quadrilateral are 2, 4, 8, 6 cent metres respectively, find the area. [Answer to the nearest s millimetre.]

2. Find the lengths of the diagonals of a cyclic quadrilatera if the sides taken in order are 3, 5, 7, 10 centimetres respe tively.

   Also find the radius of the circumscribing circle. (Answer the nearest millimetre.)

3. If 2α is the sum of two opposite angles of a quadrilater which has a circle inscribed in it, prove that the area is

$$\sqrt{abcd}\,\sin\alpha.$$

4. If a circle can be inscribed in a cyclic quadrilateral, pro that the area of the quadrilateral is $\sqrt{abcd}$, and the radius of t circle

$$2\sqrt{abcd}/(a+b+c+d).$$

5. If a circle can be inscribed in a quadrilateral, prove that the area of the quadrilateral is

$$\tfrac{1}{2}\left[x^2y^2-(ac-bd)^2\right]^{\frac{1}{2}}$$

where x and y are the diagonals.

6. The area of any quadrilateral is

$$\tfrac{1}{4}\left[4x^2y^2-(b^2+d^2-a^2-c^2)^2\right]^{\frac{1}{2}}$$

where x and y are the diagonals.

7. If ABCD is a cyclic quadrilateral, prove that

$$(s-c)(s-d)\tan^2\frac{\mathrm{B}}{2}=(s-a)(s-b).$$

8. If θ is the angle between the diagonals of a cyclic quadrilateral, prove that

$$(ac+bd)\sin\theta=2\sqrt{(s-a)(s-b)(s-c)(s-d)},$$

$$2(ac+bd)\cos\theta=(a^2+c^2)\sim(b^2+d^2).$$

9. ABCD is a cyclic quadrilateral, the circle having unit radius; α, β, γ are the angles subtended by AB, BC, CD at the circumference; prove that

$$\text{area of ABCD}=2\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta).$$

10. If 2α is the sum of two opposite angles, ϕ the angle between the diagonals, and the quadrilateral such that a circle can be inscribed in it, prove that

$$\tan^2\phi=\frac{4abcd\sin^2\alpha}{(ac-bd)^2}.$$

11. A quadrilateral is formed of four jointed rods of lengths a, b, c, d. If the area of the quadrilateral when the angle between a, b is a right angle is equal to the area when the angle between c, d is a right angle, show that either $ab=cd$, or

$$a^2+b^2=c^2+d^2.$$

12. Show that if a, b are adjacent sides of a parallelogram, α, ϕ the acute angles between the sides and between the diagonals respectively, then

$$\frac{a}{b}\sin\phi=\sin\alpha\cos\phi\pm\sqrt{1-\cos^2\alpha\cos^2\phi}.$$

13. If equilateral triangles are described on the sides of a quadrilateral outwards, and their corners joined in succession to form another quadrilateral, prove that the sum of the squares of its sides is

$$3(a^2+b^2+c^2+d^2)+4\sqrt{3}S-x^2-y^2$$

where x and y are the diagonals of the original quadrilateral.

14. If x and y are the diagonals of a quadrilateral and θ the sum of two opposite angles, prove that

$$x^2y^2=a^2c^2+b^2d^2-2abcd\cos\theta.$$

15. If a circle can be described about a quadrilateral, the ratio of the tangents drawn to the circle from the intersections of opposite sides is

$$\frac{a^2-c^2}{b^2-d^2}\sqrt{\frac{bd}{ac}}.$$

16. If it is possible to draw two circles, one touching AB, BC, CD, the other touching CD, DA, AB, and the two circles touching one another, prove that

$$(a-b+c-d)\sin\frac{A+D}{2}=4\sqrt{bd\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}.$$

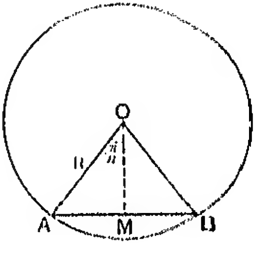
REGULAR POLYGONS.

167. *To find the area of a regular polygon of n sides and the radius of the circumscribed circle in terms of a side of the polygon.*

Let AB $(=a)$ be one of the sides of the polygon and O the centre of the circumscribing circle.

Draw OM perpendicular to AB.

$$A\hat{O}M=\tfrac{1}{2}A\hat{O}B$$
$$=\frac{1}{2}\cdot\frac{2\pi}{n}$$
$$=\frac{\pi}{n}$$
$$R=AM\operatorname{cosec}AOM$$
$$=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\ldots\ldots(i).$$



Area of polygon $= n \times$ area of AOB

$$= \frac{n}{2} \text{AB} \times \text{OM} = \frac{na}{2} \times \frac{a}{2} \cot \frac{\pi}{n}$$

$$= \frac{na^2}{4} \cot \frac{\pi}{n}.$$

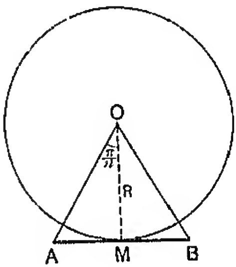
By substituting for a from (i), this value becomes

$$\tfrac{1}{2} n R^2 \sin \frac{2\pi}{n}.$$

168. *To find the area of a regular polygon of n sides and the radius of the inscribed circle in terms of a side of the polygon.*

Let AB $(= a)$ be one side of the polygon, touching the circle at M.

Join OA, OB, OM.



$$A\hat{O}M = \tfrac{1}{2} A\hat{O}B = \frac{\pi}{n},$$

$$R = \text{AM} \cot \frac{\pi}{n}$$

$$= \frac{a}{2} \cot \frac{\pi}{n} \quad \text{......(ii).}$$

Area of polygon

$$= n \times \text{area of AOB}$$

$$= \frac{n}{2} \text{AB} \times \text{OM} = \frac{na}{2} \times \frac{a}{2} \cot \frac{\pi}{n}$$

$$= \frac{na^2}{4} \cot \frac{\pi}{n}.$$

By substituting for a from (ii), this value becomes

$$n R^2 \tan \frac{\pi}{n}.$$

169. Ex. If the length of one side of a regular pentagon is 5 centimetres, find its area, and the radius of the circumscribing circle.

$$R = \frac{5}{2}\operatorname{cosec}\frac{\pi}{5} = \frac{5}{2}\operatorname{cosec} 36°,$$

$$= \frac{5}{2} \times 1{\cdot}7013 = 4{\cdot}2533 \text{ cms.}$$

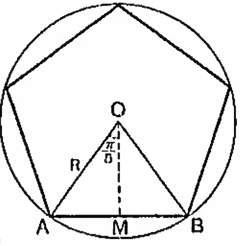
Area $= 5 \times$ area of AOB

$$= 5 \times \frac{5}{2} \text{OM}$$

$$= \frac{25}{2} \times \frac{5}{2} \cot 36°$$

$$= \frac{125}{4} \times 1{\cdot}3764 \text{ sq. cms.}$$

$= 43$ sq. cms. (to the nearest sq. cm.)



EXAMPLES XXXVIII.

1. If the length of the side of a regular hexagon is 10 centimetres, find the radius of the inscribed circle and the area of the hexagon to the nearest sq. millimetre.

2. Find the perimeter of a regular octagon which surrounds a circle of radius 2 feet. (Answer to ·001 of a foot.)

3. Find the length of the side of a regular hexagon inscribed in a circle of radius 5 centimetres.

4. Find the area of a regular decagon inscribed in a circle of 6 inches radius. (Answer to $\frac{1}{100}$ of a sq. inch.)

5. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

6. Show that the areas of the inscribed and circumscribed circles of a regular hexagon are as 3 : 4.

7. Given that a regular hexagon has an area of 200 sq. centimetres, find the area of the circle inscribed in it. (Answer to the nearest sq. centimetre.)

8. If the area of a circle is 150 sq. inches, find the area of the regular pentagon described about it. (Answer to the nearest square inch.)

9. The area of a regular hexagon is 235 sq. centimetres. Find the length of one of the sides to the nearest millimetre.

10. Two regular polygons of n sides and $2n$ sides have the same perimeter, show that their areas are as $2\cos\frac{\pi}{n} : 1 + \cos\frac{\pi}{n}$.

11. Two regular polygons of n sides are respectively circumscribed about and inscribed in a circle. Prove that their areas are as $1 : \cos^2\frac{\pi}{n}$.

12. Find the area enclosed by 200 hurdles placed so as to form a regular polygon of 200 sides, the length of each hurdle being 6 feet. (Answer to the nearest sq. foot.)

13. ABCDE is a regular pentagon. Show that if the distance of A from B or E be 34 inches, its distance from C or D will be 55 inches nearly.

Hence the general solution of

$$\tan a\theta = \tan b\mathrm{A} \quad \text{or} \quad \cot a\theta = \cot b\mathrm{A}$$

is

$$a\theta = n\pi + b\mathrm{A},$$

i.e.

$$\theta = \frac{n\pi}{a} + \frac{b}{a}\mathrm{A}.$$

173. It is interesting to notice that when an equation involves a square, the solution is always

$$n\pi \pm \mathrm{A}.$$

For if

$$\sin^2\theta = \sin^2\mathrm{A}$$

then

$$1 - \sin^2\theta = 1 - \sin^2\mathrm{A},$$

$$\therefore\ \cos^2\theta = \cos^2\mathrm{A};$$

$$\therefore\ \tan^2\theta = \tan^2\mathrm{A};$$

or if

$$\cos^2\theta = \cos^2\mathrm{A}$$

then

$$\sin^2\theta = \sin^2\mathrm{A},$$

$$\therefore\ \tan^2\theta = \tan^2\mathrm{A};$$

and thus every such equation is equivalent to

$$\tan^2\theta = \tan^2\mathrm{A},$$

$$\therefore\ \tan\theta = \tan\mathrm{A} \text{ or } \tan(-\mathrm{A}),$$

$$\therefore\ \theta = n\pi \pm \mathrm{A}.$$

ILLUSTRATIVE EXAMPLES.

174. Ex. 1. Solve

$$3\sin 7\theta - 2\sin 4\theta + 3\sin\theta = 0.$$

$$3(\sin 7\theta + \sin\theta) = 2\sin 4\theta,$$

$$6\sin 4\theta\cos 3\theta = 2\sin 4\theta.$$

$\therefore$ either $\qquad \sin 4\theta = 0,$

i.e. $\qquad 4\theta = n\pi,$

i.e. $\qquad \theta = \dfrac{n\pi}{4};$

or $\qquad \cos 3\theta = \frac{1}{3} = \dot{3} = \cos 70^\circ\ 32',$

i.e. $\qquad 3\theta = 2n\pi \pm 70^\circ\ 32'$

$$\theta = \frac{2n\pi}{3} \pm 23^\circ\ 30'\tfrac{2}{3}.$$

Ex. 2. Solve

$$\cos a\theta = \sin b\theta.$$

$$\cos a\theta = \cos\left(\frac{\pi}{2} - b\theta\right),$$

$$a\theta = 2n\pi \pm \left(\frac{\pi}{2} - b\theta\right),$$

$$\theta = \frac{(4n+1)\pi}{2(a+b)},$$

or $\qquad = \dfrac{(4n-1)\pi}{2(a-b)},$

we might have started

$$\sin\left(\frac{\pi}{2} - a\theta\right) = \sin b\theta \text{ etc.}$$

Ex. 3. Solve

$$7\cos\theta + \sin\theta = 2.$$

1st method. Change θ into $\dfrac{\theta}{2}$, thus

$$7\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right).$$

Dividing by $\cos^2\dfrac{\theta}{2}$

$$7\left(1 - \tan^2\frac{\theta}{2}\right) + 2\tan\frac{\theta}{2} = 2\left(1 + \tan^2\frac{\theta}{2}\right),$$

$$9\tan^2\frac{\theta}{2} - 2\tan\frac{\theta}{2} - 5 = 0.$$

Solving $$\tan\frac{\theta}{2} = \cdot 8647 \text{ or } -\cdot 6425,$$

$$\therefore \tan\frac{\theta}{2} = \tan 40^\circ\, 51' \text{ or } = \tan(-32^\circ\, 43'),$$

$$\therefore \frac{\theta}{2} = n\pi + 40^\circ\, 51' \text{ or } = n\pi - 32^\circ\, 43',$$

i.e. $$\theta = 2n\pi + 81^\circ\, 42' \text{ or } 2n\pi - 65^\circ\, 26'.$$

2nd method. Divide by the square root of the sum of the squares of the coefficients of $\cos\theta$ and $\sin\theta$.

$$\frac{7}{\sqrt{7^2+1^2}}\cos\theta + \frac{1}{\sqrt{7^2+1^2}}\sin\theta = \frac{2}{\sqrt{7^2+1^2}}.$$

From tables $\frac{1}{7} = \tan 8^\circ\, 8'$,

$$\therefore \frac{7}{\sqrt{7^2+1^2}} = \cos 8^\circ\, 8'; \qquad \frac{1}{\sqrt{7^2+1^2}} = \sin 8^\circ\, 8',$$

also $$\frac{2}{\sqrt{7^2+1^2}} = \cdot 2828 = \cos 73^\circ\, 34',$$

$$\therefore \cos 8^\circ\, 8' . \cos\theta + \sin 8^\circ\, 8' . \sin\theta = \cos 73^\circ\, 34',$$

$$\cos(\theta - 8^\circ\, 8') = \cos 73^\circ\, 34',$$

$$\therefore \theta - 8^\circ\, 8' = 2n\pi \pm 73^\circ\, 34',$$

$$\theta = 2n\pi + 81^\circ\, 42' \text{ or } 2n\pi - 65^\circ\, 26'.$$

Examples on this method have generally been set so as to be done by known angles; thus

$$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2},$$

$$\therefore \frac{\sqrt{3}}{2}\cos\theta + \tfrac{1}{2}\sin\theta = \frac{1}{\sqrt{2}},$$

$$\cos 30^\circ \cos\theta + \sin 30^\circ \sin\theta = \cos 45^\circ,$$

$$\cos(\theta - 30^\circ) = \cos 45^\circ,$$

$$\therefore \theta = 2n\pi \pm 45^\circ + 30^\circ = 2n\pi + \frac{5\pi}{12} \text{ or } 2n\pi - \frac{\pi}{12}.$$

Ex. 4. Solve

$$(1 - \tan\theta) = (1 - 3\tan\theta)\cos^2\theta$$

$$= \frac{(1 - 3\tan\theta)}{1 + \tan^2\theta},$$

$$\therefore\ 1 - \tan\theta + \tan^2\theta - \tan^3\theta = 1 - 3\tan\theta,$$

$\therefore$ either $\qquad \tan\theta = 0;\quad i.e.\ \theta = n\pi,$

or $\qquad \tan^2\theta - \tan\theta - 2 = 0,$

$$\therefore\ \tan\theta = 2 \text{ or } \tan\theta = -1,$$

$$\therefore\ \tan\theta = \tan 63^\circ\, 26', \qquad \text{or } \tan\theta = \tan\left(-\frac{\pi}{4}\right),$$

$$\therefore\ \theta = n\pi + 63^\circ\, 26', \qquad \text{or } \theta = n\pi - \frac{\pi}{4}.$$

Ex. 5. Solve

$$\sin^2\theta - \cos 2\theta = 1\tfrac{1}{4}.$$

$$\sin^2\theta - (1 - 2\sin^2\theta) = 1\tfrac{1}{4},$$

$$3\sin^2\theta = \tfrac{9}{4},$$

$$\sin^2\theta = \sin^2 60^\circ,$$

$$\therefore\ \theta = n\pi \pm 60^\circ. \quad \text{(Art. 173.)}$$

EXAMPLES XXXIX.

Find the general solution of

1. $\sin 2\theta = \frac{\sqrt{3}}{2}$.
2. $\cos 3\theta = \frac{1}{2}$.
3. $\tan 4\theta = 1$.
4. $\sin 5\theta = \cdot 3502$.
5. $\cos 6\theta = \cdot 95$.
6. $\tan 7\theta = \cdot 7032$.
7. $\sin^2 3\theta = \frac{3}{4}$.
8. $\cos^2 3\theta = \frac{1}{4}$.
9. $\tan^2 3\theta = 3$.
10. $\sin 2\theta = \sin\theta$.

11. $\cos 3\theta = \cos 2\theta.$

12. $\tan 4\theta = \tan 3\theta.$

13. $\sin^2 5\theta = \sin^2 \theta.$

14. $\cos^2 4\theta = \cos^2 3\theta.$

15. $\tan^2 3\theta = \tan^2 \theta.$

16. $\cos 3\theta = \sin 2\theta.$

17. $\sin 5\theta = \cos 3\theta.$

18. $\tan 7\theta = \cot 2\theta.$

19. $\sin 4\theta + \sin 2\theta = \sin 3\theta.$

20. $\cos \theta - \cos 7\theta = \sin 4\theta.$

21. $\sin 5\theta - \sin 3\theta = \frac{1}{2} \cos 4\theta.$

22. $\cos 6\theta + \cos 2\theta = (1{\cdot}4216) \cos 4\theta.$

23. $\sin 7\theta + \sin 5\theta + \sin 3\theta + \sin \theta = 0.$

24. $\cos 9\theta + \cos 7\theta - \sin 5\theta - \sin 3\theta = 0.$

25. $\sin 7\theta \sin 5\theta = \sin 3\theta \sin \theta.$

26. $\cos 9\theta \cos 7\theta = \cos 5\theta \cos 3\theta.$

27. $\sin 7\theta \cos \theta = \sin 5\theta \cos 3\theta.$

28. $\sin \theta \cos 3\theta = \sin 2\theta \cos 4\theta.$

29. $5 \cos \theta + 2 \sin \theta = 4.$

30. $8 \cos \theta + 3 \sin \theta = 5.$

31. $4 \cos \theta + 3 \sin \theta = 5.$

32. $7 \cos \theta + 2 \sin \theta = 7.$

33. $\sqrt{3} \sin \theta - \cos \theta = 1.$

34. $\sin \theta + \cos \theta = \sqrt{2}.$

35. $\sin \theta + \sqrt{3} \cos \theta = \sqrt{2}.$

36. $\sin \theta - 1 = \sqrt{3} \cos \theta.$

37. $\cos 2\theta = \cos \theta - \sin \theta.$

38. $\sin 6\theta \sin 2\theta = \frac{1}{2}.$

39. $4 \cos 3\theta + 3 \cos \theta = 0.$

40. $\tan 2\theta + 3 \cot \theta = 0.$

41. $2 \sin x - \sin 2x = 2 (1 + \cos x)^2.$

42. $\tan^2 \theta - 4 \sec \theta + 5 = 0.$

43. $\tan 2\theta = 8 \cos^2 \theta - \cot \theta.$

44. $\tan \left(\frac{\pi}{4} + \theta\right) = 3 \tan \left(\frac{\pi}{4} - \theta\right).$

45. $\dfrac{(\sin 2\theta - \cos 2\theta)}{\sqrt{2}} = 2 \sin^2 \theta - 1.$

46. $\cot 3x - 3\cot 2x + \cot x = 0$.

47. $\sin\theta + 1 = \cos\theta + \tan\theta$.

48. $8\cot\theta = \sec^2\dfrac{\theta}{2} + \operatorname{cosec}^2\dfrac{\theta}{2}$.

49. $\tan x + \tan(x+\alpha) + \tan(x+\beta) = \tan x \tan(x+\alpha)\tan(x+\beta)$.

50. $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$.

51. $2\sin^2 x + 3\cos x = 0$.

52. $3(1-\cos x) = \sin^2 x\,(3 - 2\cos x)$.

53. $\sin\left(\dfrac{\pi}{4} + \dfrac{3\theta}{2}\right) = 2\sin\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

54. $\dfrac{\cos\alpha}{\cos 2x} + \dfrac{\sin\alpha}{\sin 2x} = 2$.

55. $\sin(\alpha + x) + \sin(\beta + x) = 0$.

CHAPTER XVII.

SUBMULTIPLE ANGLES.

To express the Trigonometrical Ratios of half an angle in terms of those of the whole angle.

175. Given $\cos\alpha = k$, find $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$.

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} = \frac{1+k}{2},$$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1-k}{2};$$

$$\therefore\quad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+k}{2}};\quad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-k}{2}}.$$

It will be noticed

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}};\quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the first quadrant,

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}};\quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the second quadrant,

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}};\quad \sin\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the third quadrant,

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}\,;\quad \sin\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{2}}$$

when $\frac{\alpha}{2}$ lies in the fourth quadrant.

Considerations of the double value.

170. (i) *Arithmetical.*

Given $\cos\alpha = \cdot 6261.$

We have from Tables

$$\alpha = 51^\circ\, 14'$$

∴ also from Chap. VI,

$$\alpha = -51^\circ\, 14'$$

$$\alpha = 360^\circ - 51^\circ\, 14' = 308^\circ\, 46'$$

$$\alpha = 360^\circ + 51^\circ\, 14' = 411^\circ\, 14'$$

therefore

$$\cos\frac{\alpha}{2} = \cos\ 25^\circ\, 37' = \cdot 9017\,;\quad \sin\frac{\alpha}{2} = \sin\ 25^\circ\, 37' = \cdot 4324$$

$$\cos\frac{\alpha}{2} = \cos(-25^\circ 37') = \cdot 9017\,;\quad \sin\frac{\alpha}{2} = \sin(-25^\circ 37') = -\cdot 4324$$

$$\cos\frac{\alpha}{2} = \cos 154^\circ 23' = -\cdot 9017\,;\quad \sin\frac{\alpha}{2} = \sin(-154^\circ 23') = \cdot 4324$$

$$\cos\frac{\alpha}{2} = \cos 205^\circ\, 37' = -\cdot 9017\,;\quad \sin\frac{\alpha}{2} = \sin 205^\circ\, 37' = -\cdot 4324,$$

$$\text{i.e.}\quad \cos\frac{\alpha}{2} = \pm\cdot 9017\,;\quad \sin\frac{\alpha}{2} = \pm\cdot 4324.$$

We shall now show that these results obtained from first principles are the same as those found from Art. 175.

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} = \pm\sqrt{\frac{1+\cdot 6261}{2}} = \pm\cdot 9017$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} = \pm\sqrt{\frac{1-\cdot 6261}{2}} = \pm\cdot 4324.$$

177. (ii) *Algebraical.*

$$\cos\alpha = k = \cos A, \text{ suppose,}$$

where A is the smallest positive angle satisfying the equation, then

$$\alpha = 2n\pi \pm A.$$

$$\therefore \cos\frac{\alpha}{2} = \cos\left(n\pi \pm \frac{A}{2}\right); \quad \sin\frac{\alpha}{2} = \sin\left(n\pi \pm \frac{A}{2}\right).$$

(*a*) when n is even $= 2m$ suppose

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi \pm \frac{A}{2}\right) = \cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi \pm \frac{A}{2}\right) = \pm\sin\frac{A}{2}.$$

(*b*) when n is odd $= 2m + 1$ suppose

$$\cos\frac{\alpha}{2} = \cos\left(2m\pi + \pi \pm \frac{A}{2}\right) = -\cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2} = \sin\left(2m\pi + \pi \pm \frac{A}{2}\right) = \mp\sin\frac{A}{2}.$$

$\therefore$ for all values of n

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2}; \quad \sin\frac{\alpha}{2} = \pm\sin\frac{A}{2}.$$

$$= \pm\sqrt{\frac{1+k}{2}} \qquad = \pm\sqrt{\frac{1-k}{2}}.$$

178. (iii) *Geometrical.*

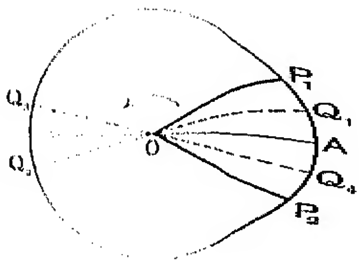
$$\cos\alpha = k = \cos A, \text{ suppose,}$$

where A is the smallest positive angle satisfying the equation.

Let $\quad A\hat{O}P_1 = A,$

$$A\hat{O}P_2 = 2\pi - A.$$

Let OP revolve positively so o trace out α, and OQ revolve itively *half as fast*, so as to o out $\frac{\alpha}{2}$.



Every time OP passes the itions OP_1, OP_2, α satisfies equation $\cos\alpha = k$, and at se moments the position of will afford us values of $\frac{\alpha}{2}$.

When OP is at OP_1 the 1st time, OQ is at OQ_1,
" " " 2nd " " OQ_2,
" " " 3rd " " OQ_1,
" " " 4th " " OQ_2,
and so on.

ain,
when OP is at OP_2 the 1st time, OQ is at OQ_3,
" " " 2nd " " OQ_4,
" " " 3rd " " OQ_3,
" " " 4th " " OQ_4,
and so on.

us

$$\cos\frac{\alpha}{2} = \cos AOQ_1 = \cos\frac{A}{2} \qquad \sin\frac{\alpha}{2} = \sin AOQ_1 = \sin\frac{A}{2}$$

$$= \cos AOQ_2 = -\cos\frac{A}{2} \qquad = \sin AOQ_2 = -\sin\frac{A}{2}$$

$$= \cos AOQ_3 = -\cos\frac{A}{2} \qquad = \sin AOQ_3 = \sin\frac{A}{2}$$

$$= \cos AOQ_4 = \cos\frac{A}{2} \qquad = \sin AOQ_4 = -\sin\frac{A}{2}.$$

us

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2} = \pm\sqrt{\frac{1+k}{2}}, \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-k}{2}}.$$

179. Given $\sin\alpha = h$; find $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$.

$$\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2 = \cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 1+h,$$

$$\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2 = \cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 1-h,$$

$$\therefore\ \cos\frac{\alpha}{2}+\sin\frac{\alpha}{2} = \pm\sqrt{1+h},$$

$$\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2} = \pm\sqrt{1-h},$$

therefore

$$\cos\frac{\alpha}{2} = \frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2}, \qquad \sin\frac{\alpha}{2} = \frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2} \quad \ldots(A),$$

$$\text{or}\quad \frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2}, \qquad \text{or}\quad \frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2} \quad \ldots(B),$$

$$\text{or} -\frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2}, \qquad \text{or} -\frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2} \quad \ldots(C),$$

$$\text{or} -\frac{\sqrt{1+h}}{2}+\frac{\sqrt{1-h}}{2}, \qquad \text{or} -\frac{\sqrt{1+h}}{2}-\frac{\sqrt{1-h}}{2} \quad \ldots(D).$$

180. Since

$$\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right) = \sqrt{2}\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right),$$

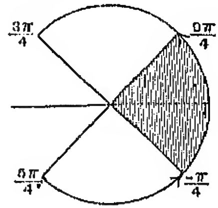
and
$$\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right) = -\sqrt{2}\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right),$$

we see that

(A) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is positive and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ negative;

i.e. when $\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is in first and second quadrants; and $\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ is in third and fourth quadrants;

i.e. when $\frac{\alpha}{2}$ lies between $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{5\pi}{4}$ and $\frac{9\pi}{4}$,



i.e. when $\frac{\alpha}{2}$ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, or generally when $\frac{\alpha}{2}$ lies between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$.

(B) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is positive and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ positive;

i.e. when $\frac{\alpha}{2}$ lies between $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$,

i.e. generally when $\frac{\alpha}{2}$ lies between $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$.

(C) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is negative and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ positive;

i.e. when $\frac{\alpha}{2}$ lies between $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$,

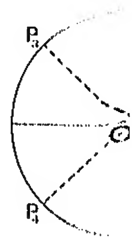
i.e. generally when $\frac{\alpha}{2}$ lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$.

(D) holds when $\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$ is negative and $\sin\left(\frac{\alpha}{2}-\frac{\pi}{4}\right)$ negative;

i.e. when $\frac{\alpha}{2}$ lies between $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$; and $\frac{\alpha}{2}$ lies between $\frac{5\pi}{4}$ and $\frac{9\pi}{4}$,

i.e. generally when $\frac{\alpha}{2}$ lies between $2n\pi+\frac{5\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$.

181. Thus from a figure



$$\cos\frac{\alpha}{2}=\frac{\sqrt{1+\sin\alpha}}{2}+\frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2}=\frac{\sqrt{1+\sin\alpha}}{2}-\frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_1OP_2,

$$\cos\frac{\alpha}{2}=\frac{\sqrt{1+\sin\alpha}}{2}-\frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2}=\frac{\sqrt{1+\sin\alpha}}{2}+\frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_2OP_3,

$$\cos\frac{\alpha}{2}=-\frac{\sqrt{1+\sin\alpha}}{2}-\frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2}=-\frac{\sqrt{1+\sin\alpha}}{2}+\frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_3OP_4,

$$\cos\frac{\alpha}{2}=-\frac{\sqrt{1+\sin\alpha}}{2}+\frac{\sqrt{1-\sin\alpha}}{2};$$

$$\sin\frac{\alpha}{2}=-\frac{\sqrt{1+\sin\alpha}}{2}-\frac{\sqrt{1-\sin\alpha}}{2},$$

when $\frac{\alpha}{2}$ lies in P_4OP_1.

This article will be found very useful in working out

Considerations of the quadruple value.

182. (i) *Arithmetical.*

Given $\sin\alpha = \cdot 7797$.

have from Tables

$$\alpha = 51^\circ\, 14',$$

from Chap. VI

$$\alpha = 180^\circ - 51^\circ\, 14' = 128^\circ\, 46',$$
$$\alpha = 360^\circ + 51^\circ\, 14' = 411^\circ\, 14',$$
$$\alpha = 540^\circ - 51^\circ\, 14' = 488^\circ\, 46'.$$

$$\begin{aligned} \cos\frac{\alpha}{2} &= \cos\ 25^\circ\, 37' = \cdot 9017; & \sin\frac{\alpha}{2} &= \sin\ 25^\circ\, 37' = \cdot 4324 \\ &= \cos\ 64^\circ\, 23' = \cdot 4324; & &= \sin\ 64^\circ\, 23' = \cdot 9017 \\ &= \cos 205^\circ\, 37' = -\cdot 9017; & &= \sin 205^\circ\, 37' = -\cdot 4324 \\ &= \cos 244^\circ\, 23' = -\cdot 4324; & &= \sin 244^\circ\, 23' = -\cdot 9017. \end{aligned}$$

We shall now show that these results obtained from first nciples are the same as those found from Art. 179.

$$\frac{\sqrt{1+\cdot 7797}}{2} + \frac{\sqrt{1-\cdot 7797}}{2} = \cdot 9017,$$

$$\frac{\sqrt{1+\cdot 7797}}{2} - \frac{\sqrt{1-\cdot 7797}}{2} = \cdot 4324.$$

183. (ii) *Algebraical.*

$$\sin\alpha = h = \sin A, \text{ suppose},$$

ere A is the smallest positive angle satisfying the equation.

en $$\alpha = n\pi + (-1)^n A.$$

$$\therefore\ \cos\frac{\alpha}{2} = \cos\left\{\frac{n\pi}{2} + (-1)^n\frac{A}{2}\right\};$$

$$\sin\frac{\alpha}{2} = \sin\left\{\frac{n\pi}{2} + (-1)^n\frac{A}{2}\right\};$$

(*a*) when n is of form $4m$

$$\cos\frac{\alpha}{2}=\cos\left(2m\pi+\frac{A}{2}\right)=\cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2}=\sin\left(2m\pi+\frac{A}{2}\right)=\sin\frac{A}{2}.$$

(*b*) when n is of form $4m+1$

$$\cos\frac{\alpha}{2}=\cos\left(2m\pi+\frac{\pi}{2}-\frac{A}{2}\right)=\sin\frac{A}{2};$$

$$\sin\frac{\alpha}{2}=\sin\left(2m\pi+\frac{\pi}{2}-\frac{A}{2}\right)=\cos\frac{A}{2}.$$

(*c*) when n is of form $4m+2$

$$\cos\frac{\alpha}{2}=\cos\left(2m\pi+\pi+\frac{A}{2}\right)=-\cos\frac{A}{2};$$

$$\sin\frac{\alpha}{2}=\sin\left(2m\pi+\pi+\frac{A}{2}\right)=-\sin\frac{A}{2}.$$

(*d*) when n is of form $4m+3$

$$\cos\frac{\alpha}{2}=\cos\left(2m\pi+\pi+\frac{\pi}{2}-\frac{A}{2}\right)=-\sin\frac{A}{2};$$

$$\sin\frac{\alpha}{2}=\sin\left(2m\pi+\pi+\frac{\pi}{2}-\frac{A}{2}\right)=-\cos\frac{A}{2}.$$

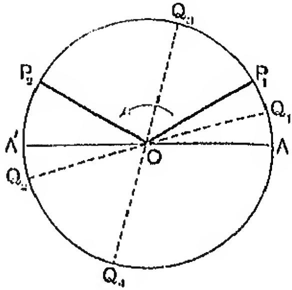
Therefore for all values of n

$$\cos\frac{\alpha}{2}=\pm\cos\frac{A}{2},\quad\text{or}\ \pm\sin\frac{A}{2};\qquad \sin\frac{\alpha}{2}=\pm\sin\frac{A}{2},\quad\text{or}\ \pm\cos\frac{A}{2}.$$

184. (iii) *Geometrical.*

$$\sin\alpha=h=\sin A,\ \text{suppose},$$

where A is the smallest positive angle satisfying the equation.



$OP_1 = A$; $AOP_2 = \pi - A$.

t OP revolve positively so trace out α, and OQ revolve vely *half as fast*, so as to out $\frac{\alpha}{2}$.

very time OP passes the ons OP_1, OP_2, α satisfies the ion $\sin\alpha = k$; and at these nts the position of OQ will us values of $\frac{\alpha}{2}$.

When OP is at OP_1	the	1st	time,	OQ	is at	OQ_1,
"	"	"	2nd	"	"	OQ_2,
"	"	"	3rd	"	"	OQ_1,
"	"	"	4th	"	"	OQ_2,

and so on.

1,

when OP is at OP_2	the	1st	time,	OQ	is at	OQ_3,
"	"	"	2nd	"	"	OQ_4,
"	"	"	3rd	"	"	OQ_3,
"	"	"	4th	"	"	OQ_4,

and so on.

$$\cos\frac{\alpha}{2} = \cos AOQ_1 = \cos\frac{A}{2}$$

$$= \cos AOQ_2 = -\cos\frac{A}{2}$$

$$= \cos AOQ_3 = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2}$$

$$= \cos AOQ_4 = -\sin\frac{A}{2}.$$

Hence

$$\cos\frac{\alpha}{2} = \pm\cos\frac{A}{2} = \pm\left(\frac{\sqrt{1+h}}{2} + \frac{\sqrt{1-h}}{2}\right)$$

$$\text{or } \pm\sin\frac{A}{2} \text{ or } \pm\left(\frac{\sqrt{1+h}}{2} - \frac{\sqrt{1-h}}{2}\right).$$

$$\sin\frac{\alpha}{2} = \sin AOQ_1 = \sin\frac{A}{2}$$

$$= \sin AOQ_2 = -\sin\frac{A}{2}$$

$$= \sin AOQ_3 = \sin\left(\frac{\pi}{2} - \frac{A}{2}\right) = \cos\frac{A}{2}$$

$$= \sin AOQ_4 = -\cos\frac{A}{2}.$$

Hence

$$\sin\frac{\alpha}{2} = \pm\sin\frac{A}{2} = \pm\left(\frac{\sqrt{1+h}}{2} - \frac{\sqrt{1-h}}{2}\right),$$

$$\text{or } \pm\cos\frac{A}{2} \text{ or } \pm\left(\frac{\sqrt{1+h}}{2} + \frac{\sqrt{1-h}}{2}\right).$$

185. Given $\tan\alpha = k$, find $\tan\frac{\alpha}{2}$.

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = k,$$

$$\therefore\ \tan^2\frac{\alpha}{2} + \frac{2}{k}\tan\frac{\alpha}{2} - 1 = 0,$$

$$\therefore\ \tan\frac{\alpha}{2} = \frac{-1\pm\sqrt{1+k^2}}{k}.$$

Thus
$$\tan\frac{\alpha}{2} = \frac{-1+\sqrt{1+\tan^2\alpha}}{\tan\alpha},$$

when $\frac{\alpha}{2}$ lies in the first and third quadrants,

$$\tan\frac{\alpha}{2}=\frac{-1-\sqrt{1+\tan^2\alpha}}{\tan\alpha},$$

when $\frac{\alpha}{2}$ lies in the second and fourth quadrants.

36. Algebraical consideration of the double
!.

$$\tan\alpha=k=\tan\mathrm{A},\ \text{suppose},$$

ι A is the smallest positive angle satisfying the equation,

$$\alpha=n\pi+\mathrm{A}.$$

ιse (i) n even $=2m$,

$$\frac{\alpha}{2}=m\pi+\frac{\mathrm{A}}{2};$$

$$\therefore\ \tan\frac{\alpha}{2}=\tan\frac{\mathrm{A}}{2}.$$

ιse (ii) n odd $=2m+1$,

$$\frac{\alpha}{2}=m\pi+\frac{\pi}{2}+\frac{\mathrm{A}}{2};$$

$$\therefore\ \tan\frac{\alpha}{2}=\tan\left(\frac{\pi}{2}+\frac{\mathrm{A}}{2}\right)=-\cot\frac{\mathrm{A}}{2}.$$

ıe arithmetical and geometrical considerations are left exercise for the student.

ILLUSTRATIVE EXAMPLES.

!7. Ex. 1. Prove that

$$2\cos\frac{\theta}{2}=-\sqrt{1-\sin\theta}-\sqrt{1+\sin\theta},$$

θ lies between 270° and 450°.

nce $\frac{\theta}{2}$ lies between 135° and 225°, *i.e.* in P_8OP_4 (Art. 181),

$$2\cos\frac{\theta}{2}=-\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}.$$

Ex. 2. Show that

$$\cos\frac{\pi}{16}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}};\ \sin\frac{\pi}{16}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}.$$

From Art. 175, $\cos\frac{\pi}{8}=+\sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2}}.$

$$\cos\frac{\pi}{16}=+\sqrt{\frac{1+\cos\frac{\pi}{8}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}.$$

Also

$$\sin\frac{\pi}{16}=+\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}.$$

Ex. 3. Given that $\sin 210°=-\frac{1}{2}$, find the values of $\sin 105°$ and $\cos 105°$.

Since $105°$ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, we have by Art. 181,

$$\cos 105°=\frac{\sqrt{1+\sin 210°}}{2}-\frac{\sqrt{1-\sin 210°}}{2}$$

$$=\frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}}=\frac{1-\sqrt{3}}{2\sqrt{2}},$$

$$\sin 105°=\frac{\sqrt{1+\sin 210°}}{2}+\frac{\sqrt{1-\sin 210°}}{2}$$

$$=\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}=\frac{1+\sqrt{3}}{2\sqrt{2}}.$$

EXAMPLES XL.

Prove that

1. $$2\cos\frac{\theta}{2}=\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta},$$

$$2\sin\frac{\theta}{2}=\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta},$$

when θ lies between $630°$ and $810°$ or between $-810°$ and $-630°$.

2. $2\cos\frac{\theta}{2} = \sqrt{1+\sin\theta} - \sqrt{1-\sin\theta}$,

$$2\sin\frac{\theta}{2} = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta},$$

when θ lies between 810° and 990° or between −630° and −450° or between 90° and 270°.

3. $2\cos\frac{\theta}{2} = -\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta}$,

$$2\sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta},$$

when θ lies between 990° and 1170° or between −450° and −270° or between 270° and 450°.

4. $2\cos\frac{\theta}{2} = -\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$,

$$2\sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta},$$

when θ lies between 1170° and 1350° or between −270° and −90° or between 450° and 630°.

5. If $\theta = 200°, 400°, 600°, 800°, 1100°$, show that

$$\tan\theta = \frac{-1+\sqrt{1+\tan^2 2\theta}}{\tan 2\theta}.$$

6. If $\theta = 100°, 300°, 500°, 700°, 1000°$, show that

$$\tan\theta = \frac{-1-\sqrt{1+\tan^2 2\theta}}{\tan 2\theta}.$$

7. If $\cos A = \frac{7}{25}$, find $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$, A being between 270° and 360°.

8. If $\sin A = \frac{240}{289}$ and A lie between 270° and 450°, find the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

9. Having given that $\sin 260° = -0{\cdot}9848$, find the values of $\sin 130°$ and $\cos 130°$.

10. Find $\sin 115°$ and $\cos 115°$, given that

$$\cos 230° = -0{\cdot}6428.$$

11. If $\tan 2A = \frac{24}{7}$, find the values of $\tan A$.

Prove that

12. $\sin\frac{\pi}{12} = \frac{1}{2}\sqrt{2-\sqrt{3}}$.

13. $\cos\frac{\pi}{24} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}$.

14. $\sin\frac{\pi}{20} = \frac{1}{2}\sqrt{2-\sqrt{\frac{1}{2}(5+\sqrt{5})}} = \frac{1}{8}\left[\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}}\right]$.

15. $\tan\frac{\pi}{8} = \sqrt{2}-1$.

16. If $\cos 4\theta = a$, the possible values of $\tan\theta$ are the four values of

$$\frac{\sqrt{2}\pm\sqrt{1+a}}{\pm\sqrt{1-a}}.$$

17. Prove that $\sin\frac{\pi}{4}\operatorname{cosec}\frac{\pi}{12} + 4\sin\frac{\pi}{10} = \sqrt{3}+\sqrt{5}$.

CHAPTER XVIII.

INVERSE CIRCULAR FUNCTIONS.

188. **Def.** $\sin^{-1}x$ stands for "The numerically smallest angle whose sine is x."

$\cos^{-1}x$ stands for "The numerically smallest angle whose cosine is x," etc.

Rule. When there are two numerically smallest angles take the positive one,

e.g. $\cos^{-1}\frac{1}{2} = +60^\circ$ and not -60°.

N.B. $\sin^{-1}(\sin x) = \text{angle } x$; $\cos^{-1}(\cos x) = \text{angle } x$; $\tan^{-1}(\tan x) = \text{angle } x$, etc.

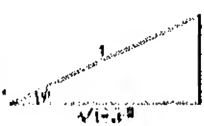
These equations put into words are seen to require no proof.

Some writers regard $\sin^{-1}x$ as many-valued; thus $\sin^{-1}x$ would equal $n\pi + (-1)^n \sin^{-1}x$; but in elementary work the student is advised to consider the above value only which is sometimes called The Principal Value.

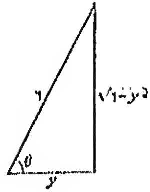
189. From a figure the student at once sees

$$\phi = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

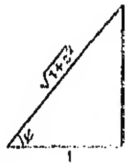
$$= \operatorname{cosec}^{-1}\frac{1}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}.$$



$$\theta = \cos^{-1} y = \sin^{-1} \sqrt{1-y^2}$$
$$= \tan^{-1} \frac{\sqrt{1-y^2}}{y}$$
$$= \cot^{-1} \frac{y}{\sqrt{1-y^2}}$$
$$= \sec^{-1} \frac{1}{y}$$
$$= \operatorname{cosec}^{-1} \frac{1}{\sqrt{1-y^2}},$$



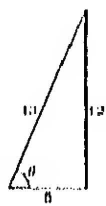
$$\psi = \tan^{-1} z = \sin^{-1} \frac{z}{\sqrt{1+z^2}}$$
$$= \cos^{-1} \frac{1}{\sqrt{1+z^2}}$$
$$= \cot^{-1} \frac{1}{z}$$
$$= \sec^{-1} \sqrt{1+z^2}$$
$$= \operatorname{cosec}^{-1} \frac{\sqrt{1+z^2}}{z}.$$



The above values need not be remembered, a figure at once recalls them.

190. A numerical example will make the above more clear. From the figure we see at once

$$\theta = \sin^{-1} \frac{12}{13} = \cos^{-1} \frac{5}{13} = \tan^{-1} \frac{12}{5} \text{ etc.}$$



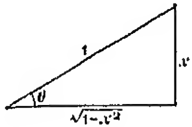
The addition and subtraction of inverse functions.

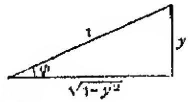
191. (i) *Find the value of*

$$\sin^{-1} x \pm \sin^{-1} y; \quad \textit{and of} \quad 2\sin^{-1} x.$$

Draw two figures putting

$$\sin^{-1} x = \theta\,; \quad \sin^{-1} y = \phi.$$





$$\sin^{-1} x + \sin^{-1} y = \theta + \phi$$

$$\begin{array}{l|l} = \sin^{-1}\{\sin(\theta+\phi)\} & = \cos^{-1}\{\cos(\theta+\phi)\}\,; \text{ etc.} \\ = \sin^{-1}\{\sin\theta\cos\phi + \sin\phi\cos\theta\} & = \cos^{-1}\{\cos\theta\cos\phi - \sin\theta\sin\phi\} \\ = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} & = \cos^{-1}\{\sqrt{1-x^2}\sqrt{1-y^2} - xy\}\,; \end{array}$$

obviously

$$\begin{aligned} \sin^{-1} x - \sin^{-1} y &= \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\} \\ &= \cos^{-1}\{\sqrt{1-x^2}\sqrt{1-y^2} + xy\}. \end{aligned}$$

192. $\sin^{-1} x \pm \sin^{-1} y = \theta \pm \phi = \tan^{-1}\{\tan(\theta \pm \phi)\}$

$$= \tan^{-1}\left\{\frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}\right\}$$

$$= \tan^{-1}\left\{\frac{\dfrac{x}{\sqrt{1-x^2}} \pm \dfrac{y}{\sqrt{1-y^2}}}{1 \mp \dfrac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}}\right\}.$$

193. $2\sin^{-1} x = 2\theta = \sin^{-1}(\sin 2\theta) = \cos^{-1}(\cos 2\theta)$; etc.

$$\begin{array}{l|l} = \sin^{-1}(2\sin\theta\cos\theta) & = \cos^{-1}(1 - 2\sin^2\theta) \\ = \sin^{-1}(2x\sqrt{1-x^2}) & = \cos^{-1}(1 - 2x^2). \end{array}$$

We leave it for the student to show

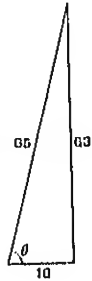
$$\begin{aligned} \cos^{-1} x \pm \cos^{-1} y &= \cos^{-1}\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\} \\ &= \sin^{-1}\{y\sqrt{1-x^2} \pm x\sqrt{1-y^2}\} \text{ etc.} \end{aligned}$$

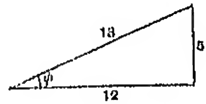
and $$2\cos^{-1} x = \cos^{-1}(2x^2 - 1) = \sin^{-1}(2x\sqrt{1-x^2}).$$

Draw two figures putting

$$\cos^{-1}\tfrac{16}{65}=\theta;\quad \cos^{-1}\tfrac{12}{13}=\phi.$$

$$\cos^{-1}\tfrac{16}{65}-\cos^{-1}\tfrac{12}{13}=\theta-\phi$$
$$=\sin^{-1}\{\sin(\theta-\phi)\}$$
$$=\sin^{-1}\{\sin\theta\cos\phi-\sin\phi\cos\theta\}$$
$$=\sin^{-1}\left\{\frac{63}{65}\cdot\frac{12}{13}-\frac{5}{13}\cdot\frac{16}{65}\right\}$$
$$=\sin^{-1}\left(\frac{4\,.\,169}{65\,.\,13}\right)=\sin^{-1}\tfrac{4}{5}.$$





This example should be verified from Tables; thus

$$\cos^{-1}\tfrac{16}{65}=\cos^{-1}(\cdot 2462)=75^\circ\,45',$$
$$\cos^{-1}\tfrac{12}{13}=\cos^{-1}(\cdot 9231)=22^\circ\,37',$$
$$\sin^{-1}\tfrac{4}{5}=\sin^{-1}(\cdot 8)\quad=53^\circ\,8',$$

and

$$75^\circ\,45'-22^\circ\,37'=53^\circ\,8'.$$

Ex. 2. Prove

$$\tan^{-1}\tfrac{2}{11}+2\tan^{-1}\tfrac{1}{7}=\tan^{-1}\tfrac{1}{2}.$$

Call $\tan^{-1}\tfrac{2}{11}=\theta;\quad \tan^{-1}\tfrac{1}{7}=\phi;$

then $\tan^{-1}\tfrac{2}{11}+2\tan^{-1}\tfrac{1}{7}=\theta+2\phi=\tan^{-1}\{\tan(\theta+2\phi)\}$

$$=\tan^{-1}\left\{\frac{\tan\theta+\tan 2\phi}{1-\tan\theta\tan 2\phi}\right\}$$
$$=\tan^{-1}\left\{\frac{\tan\theta+\dfrac{2\tan\phi}{1-\tan^2\phi}}{1-\tan\theta\,.\,\dfrac{2\tan\phi}{1-\tan^2\phi}}\right\}$$
$$=\tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}}\right\}$$
$$=\tan^{-1}\left(\frac{48+77}{264-14}\right)=\tan^{-1}\tfrac{1}{2}.$$

This should be verified from Tables, thus

$$\tan^{-1}\tfrac{2}{11}=\tan^{-1}(\cdot 1818)=10^\circ\,18',$$
$$2\tan^{-1}\tfrac{1}{7}=2\tan^{-1}(\cdot 1429)=2(8^\circ\,8')=16^\circ\,16',$$
$$\tan^{-1}\tfrac{1}{2}=\tan^{-1}(\cdot 5)\quad=26^\circ\,34',$$

and

$$10^\circ\,18'+16^\circ\,16'=26^\circ\,34'.$$

EXAMPLES XLI.

Complete the following:

1. $\sin^{-1}\frac{12}{13} = \tan^{-1}[\qquad]$.

2. $\cos^{-1}\frac{3}{5} = \cot^{-1}[\qquad]$.

3. $\tan^{-1}\frac{63}{16} = \sin^{-1}[\qquad]$.

4. $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}[\qquad]$.

Prove that

5. $\cos^{-1}\frac{63}{65} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{4}{5}$.

6. $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{63}{16}$.

7. $2\cos^{-1}\frac{3}{5} = \cos^{-1}(-\frac{7}{25})$.

8. $2\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{24}{25}$.

9. $2\tan^{-1}\frac{2}{3} = \tan^{-1}\frac{12}{5}$.

10. $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{17}{19}$.

11. $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{32}{43}$.

12. $\tan^{-1}\frac{3}{8} + \tan^{-1}\frac{2}{9} = \tan^{-1}\frac{43}{66}$.

13. $\tan^{-1}\frac{5}{2} + \tan^{-1}\frac{3}{4} = \tan^{-1}(-\frac{26}{7})$.

14. $\tan^{-1}\frac{7}{4} - \tan^{-1}\frac{1}{6} = \tan^{-1}\frac{38}{31}$.

15. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^\circ$.

16. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = 45^\circ$.

17. $2\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} = \frac{\pi}{4} - \tan^{-1}\frac{6}{61}$.

18. $\tan^{-1}n + \cot^{-1}(n+1) = \tan^{-1}(n^2+n+1)$.

19. $\tan^{-1}\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right) - \tan^{-1}\left(\dfrac{x-\cos\alpha}{\sin\alpha}\right) = \dfrac{\pi}{2} - \alpha$.

20. $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$.

21. $\cos^{-1}\dfrac{a-x}{a+x} = 2\tan^{-1}\sqrt{\dfrac{x}{a}}$.

(ii) Solve

$$\sin^{-1} 2x = \sin^{-1} x\sqrt{3} + \sin^{-1} x.$$

From Art. 191,

$$\sin^{-1} 2x = \sin^{-1}\{x\sqrt{3}\sqrt{1-x^2} + x\sqrt{1-3x^2}\},$$

$$\therefore\ 2x = x\sqrt{3}\sqrt{1-x^2} + x\sqrt{1-3x^2},$$

therefore either $\qquad x = 0,$

or $\qquad 2 = \sqrt{3}\sqrt{1-x^2} + \sqrt{1-3x^2},$

i.e. $\qquad 4 + (1-3x^2) - 4\sqrt{1-3x^2} = 3(1-x^2),$

$$\therefore\ 4\sqrt{1-3x^2} = 2,$$

$$1 - 3x^2 = \tfrac{1}{4},$$

$$\therefore\ x^2 = \tfrac{1}{4}, \quad \therefore\ x = \pm\tfrac{1}{2}.$$

Ans. $0;\ \pm\frac{1}{2}$.

EXAMPLES XLII.

Solve

1. $\tan^{-1} 2x + \tan^{-1} 3x = \dfrac{\pi}{4}.$

2. $\tan^{-1}\dfrac{1}{a-1} = \tan^{-1}\dfrac{1}{x} + \tan^{-1}\dfrac{1}{a^2 - x + 1}.$

3. $\tan^{-1}\dfrac{x+1}{x-1} + \tan^{-1}\dfrac{x-1}{x} = \tan^{-1}(-7).$

4. $\tan^{-1}(ax+b) + \tan^{-1}(ax-b) = \dfrac{\pi}{4}.$

5. $\sin^{-1} x + \sin^{-1}\dfrac{x}{2} = \dfrac{\pi}{4}.$

6. $\operatorname{cosec}^{-1} x = \operatorname{cosec}^{-1} a + \operatorname{cosec}^{-1} b.$

7. $\cos^{-1}\dfrac{1}{\sqrt{1+x^2}} - \cos^{-1}\dfrac{x}{\sqrt{1+x^2}} = \sin^{-1}\dfrac{1+x}{1+x^2}.$

8. $\sin^{-1}\dfrac{5}{x}+\sin^{-1}\dfrac{12}{x}=\dfrac{\pi}{2}$.

9. Solve $\tan^{-1}\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=30^\circ$.

10. $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$.

11. $\sin^{-1}x+\sin^{-1}\frac{1}{2}=\sin^{-1}\frac{3}{4}$.

12. $\sin^{-1}\dfrac{12}{13}+\sin^{-1}\dfrac{12}{x}=\dfrac{\pi}{2}$.

13. $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$.

14. $\cot^{-1}x-\cot^{-1}(x+2)=15^\circ$.

15. $\sec^{-1}\dfrac{x}{a}-\sec^{-1}\dfrac{x}{b}=\sec^{-1}b-\sec^{-1}a$.

16. $\cot^{-1}(x-a)+\cot^{-1}(x-b)+\cot^{-1}(x-c)=0$.

17. If $\tan^{-1}a=\operatorname{cosec}^{-1}a=\cos^{-1}b$, prove that one value of b is $\dfrac{\sqrt{5}-1}{2}$.

CHAPTER XIX.

ELIMINATION.

197. FROM certain equations it is frequently desirable to deduce others which shall not contain certain variables. This process is called Elimination and the result obtained the **Eliminant.**

If the number of equations given is one greater than the number of variables, it is always possible to eliminate those variables.

Each problem must be considered on its own merits.

Ex. 1. Eliminate θ between

$$a \cos\theta + b \sin\theta = c,$$

and

$$a' \cos\theta + b' \sin\theta = c'.$$

Solving for $\cos\theta$ and $\sin\theta$, we obtain

$$\cos\theta = \frac{bc' - b'c}{ba' - b'a},$$

$$\sin\theta = \frac{a'c - ac'}{ba' - b'a}.$$

$\therefore$ squaring and adding,

$$1 = \left(\frac{bc' - b'c}{ba' - b'a}\right)^2 + \left(\frac{a'c - ac'}{ba' - b'a}\right)^2,$$

or

$$(ba' - b'a)^2 = (bc' - b'c)^2 + (a'c - ac')^2.$$

Ex. 2. Eliminate θ between

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \text{.................(i),}$$

and

$$\tan\theta = c \quad \text{.......................(ii).}$$

From (ii)

$$\frac{\sin\theta}{c} = \frac{\cos\theta}{1} = \sqrt{\frac{\sin^2\theta + \cos^2\theta}{c^2+1}}$$

$$= \frac{1}{\sqrt{c^2+1}},$$

$\therefore$ substituting in (i)

$$a\sqrt{c^2+1}\,x - \frac{b\sqrt{c^2+1}}{c}\,y = a^2 - b^2.$$

Ex. 3. Eliminate θ between

$$\frac{\sec^4\phi - 1}{\sec^4\phi + 1} = \frac{x}{a} \quad \text{.......................(i),}$$

and

$$\sec^2\phi + \cos^2\phi = \frac{2b}{y} \quad \text{.......................(ii).}$$

From (ii)

$$\frac{b}{y} = \frac{\sec^4\phi + 1}{2\sec^2\phi},$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(\sec^4\phi - 1)^2 + 4\sec^4\phi}{(\sec^4\phi + 1)^2}$$

$$= \frac{(\sec^4\phi + 1)^2}{(\sec^4\phi + 1)^2} = 1.$$

Ex. 4. Eliminate θ and ϕ between

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{...................(i),}$$

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1 \quad \text{...................(ii),}$$

$$\theta - \phi = 2a \quad \text{................(iii).}$$

Solving for $\frac{x}{a}$ and $\frac{y}{b}$ from (i) and (ii)

$$\frac{\frac{x}{a}}{\sin\theta - \sin\phi} = \frac{\frac{y}{b}}{\cos\phi - \cos\theta} = \frac{1}{\sin(\theta-\phi)} = \frac{1}{\sin 2a},$$

$$\therefore \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{(\sin\theta - \sin\phi)^2 + (\cos\phi - \cos\theta)^2}{\sin^2 2a}$$
$$= \frac{2 - 2(\sin\theta\sin\phi + \cos\theta\cos\phi)}{\sin^2 2a}$$
$$= \frac{2 - 2\cos 2a}{\sin^2 2a}$$
$$= \frac{4\sin^2 a}{4\sin^2 a\cos^2 a} = \frac{1}{\cos^2 a}.$$

Ex. 5. Eliminate θ and ϕ between

$$x\cos\theta + y\sin\theta = x\cos\phi + y\sin\phi = 1,$$

and
$$a\cos\theta\cos\phi + b\sin\theta\sin\phi + c + g(\cos\theta + \cos\phi)$$
$$+ f(\sin\theta + \sin\phi) + h\sin(\theta + \phi) = 0.$$

From
$$x\cos\theta + y\sin\theta = 1,$$
$$x\cos\phi + y\sin\phi = 1,$$

we obtain $\dfrac{\cos\frac{1}{2}(\theta+\phi)}{x} = \dfrac{\sin\frac{1}{2}(\theta+\phi)}{y} = \dfrac{\cos\frac{1}{2}(\theta-\phi)}{1}$ ......(i),

$\therefore$ Each fraction $= \dfrac{\sin\frac{1}{2}(\theta-\phi)}{\sqrt{x^2+y^2-1}}$ and also $= \dfrac{1}{\sqrt{x^2+y^2}}$ ....(ii).

The third equation may be written

$$a[\cos(\theta+\phi) + \cos(\theta-\phi)] + b[\cos(\theta-\phi) - \cos(\theta+\phi)]$$
$$+ 2c + 4g[\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)] + 4f[\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)]$$
$$+ 4h\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta+\phi) = 0 \quad \text{.............................(iii).}$$

From (i) and (ii),

$$\cos(\theta+\phi) = 2\cos^2\tfrac{1}{2}(\theta+\phi) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2},$$
$$\cos(\theta-\phi) = 2\cos^2\tfrac{1}{2}(\theta-\phi) - 1 = \frac{2}{x^2+y^2} - 1 = \frac{2-x^2-y^2}{x^2+y^2}.$$

Therefore, substituting in (iii)

$$a\frac{x^2-y^2+2-x^2-y^2}{x^2+y^2} + b\frac{2-x^2-y^2-x^2+y^2}{x^2+y^2} + 2c + 4g\frac{x}{x^2+y^2}$$
$$+ 4f\frac{y}{x^2+y^2} + 4h\frac{xy}{x^2+y^2} = 0,$$

$$a(1-y^2) + b(1-x^2) + c(x^2+y^2) + 2gx + 2fy + 2hxy = 0.$$

EXAMPLES XLIII.

Eliminate ϕ between the equations:

1. $x = a\cos\phi, \quad y = b\sin\phi.$

2. $x = \sin\phi - \operatorname{cosec}\phi, \quad y = \cos\phi - \sec\phi.$

3. $\dfrac{\cos\phi}{h} = \dfrac{\sin\phi}{k} = \dfrac{a\cos\phi + b}{a^2}.$

4. $\sin\theta = a\cos\phi + b\sin\phi, \quad \cos\theta = a\sin\phi - b\cos\phi.$

5. $x = \sin\phi + \cos\phi, \quad y = \tan\phi + \cot\phi.$

6. $x = \sin\phi + \tan\phi, \quad y = \sin\phi - \tan\phi.$

7. $x\sin\phi - y\cos\phi = \sqrt{x^2 + y^2}, \quad \dfrac{\sin^2\phi}{a^2} + \dfrac{\cos^2\phi}{b^2} = \dfrac{1}{x^2 + y^2}.$

8. $x = a\cot^2\phi, \quad y = 2a\tan\phi.$

9. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1, \quad -\dfrac{x}{a}\sin\phi + \dfrac{y}{b}\cos\phi = 1.$

10. $x = 3\cos\phi + \cos 3\phi, \quad y = 3\sin\phi - \sin 3\phi.$

11. $x = a\cos\phi\,(4\cos^2\phi - 3), \quad y = b\sin\phi\,(4\cos^2\phi - 1).$

12. $\dfrac{x}{a}\cos\phi + \dfrac{y}{b}\sin\phi = 1, \quad x\sin\phi - y\cos\phi$
$= (a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}}.$

Eliminate θ and ϕ from:

13. $a\sin\theta - b\sin\phi = 0, \quad c\cos\theta - d\cos\phi = 0, \quad \theta - 2\phi = 0.$

14. $\tan\theta + \tan\phi = a, \quad \cot\theta + \cot\phi = b, \quad \theta + \phi = \alpha.$

15. $\tan\theta + \tan\phi = a, \quad \cot\theta + \cot\phi = b, \quad \theta - \phi = \alpha.$

16. $\cos\theta + \cos\phi = a, \quad \cot\theta + \cot\phi = b, \quad \operatorname{cosec}\theta + \operatorname{cosec}\phi = c.$

17. $\sin\alpha\cos\theta = \sin\beta, \quad \sin\alpha\cos\phi = \sin\gamma, \quad \cos(\theta - \phi)$
$= \sin\beta\sin\gamma.$

18. $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2 = \dfrac{ax}{\cos\phi} - \dfrac{by}{\sin\phi}, \quad \theta - \phi = \dfrac{\pi}{2}.$

19. $\sin\theta + \sin\phi = a, \quad \cos\theta + \cos\phi = b, \quad \tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2} = c.$

Eliminate θ between:

20. $a\sin\theta + b\tan\theta = m, \quad a\cos\theta + b\cot\theta = n.$

21. $x\cos\theta + y\sin\theta = a\cos 2\theta, \quad x\sin\theta - y\cos\theta = 2a\sin 2\theta.$

22. $\operatorname{cosec}\theta - \sin\theta = m, \quad \sec\theta - \cos\theta = n.$

23. $x\cos(\theta + a) + y\sin(\theta + a) = a\sin 2\theta,$
$y\cos(\theta + a) - x\sin(\theta + a) = 2a\cos 2\theta.$

24. $(a + b)\tan(\theta - \phi) = (a - b)\tan(\theta + \phi),$
$a\cos 2\phi + b\cos 2\theta = c.$

25. $x = 2a\cos\theta + a\cos 2\theta,$
$y = 2a\sin\theta - a\sin 2\theta.$

CHAPTER XX.

INEQUALITIES AND LIMITS.

Throughout this chapter

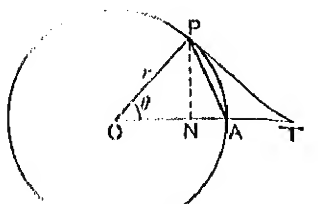
θ = the *Circular Measure* of a positive *Acute* angle.

198. To show $\tan\theta > \theta > \sin\theta$.

Draw a circle radius r, centre O, and let

$$A\hat{O}P = \theta.$$

Draw PT a tangent and PN perpendicular to OA.



Then from fig.

OPT > sector AOP > △ AOP,

$$\therefore \tfrac{1}{2}OP \cdot PT > \tfrac{1}{2}r^2\theta$$

$$> \tfrac{1}{2}PN \cdot OA,$$

$$\therefore \tfrac{1}{2}r \cdot r\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r\sin\theta \cdot r,$$

i.e. $\tan\theta > \theta > \sin\theta$.

199. The limiting values of

$$\frac{\sin\theta}{\theta} \text{ and } \frac{\tan\theta}{\theta},$$

as θ is indefinitely diminished, are each **unity**.

By previous art.

$$\tan\theta > \theta > \sin\theta;$$

$$\therefore \quad \sec\theta > \frac{\theta}{\sin\theta} > 1.$$

But in the limit when θ is indefinitely diminished

$$\sec\theta = 1,$$

hence
$$\underset{\theta=0}{\mathrm{Lt}} \frac{\theta}{\sin\theta} = 1,$$

and therefore
$$\underset{\theta=0}{\mathrm{Lt}} \frac{\sin\theta}{\theta} = 1;$$

again
$$\tan\theta > \theta > \sin\theta;$$

$$\therefore \quad 1 > \frac{\theta}{\tan\theta} > \cos\theta.$$

But in the limit when θ is indefinitely diminished

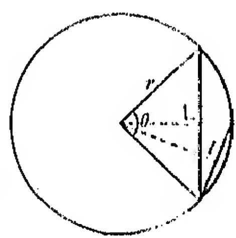
$$\cos\theta = 1,$$

hence
$$\underset{\theta=0}{\mathrm{Lt}} \frac{\theta}{\tan\theta} = 1,$$

and therefore
$$\underset{\theta=0}{\mathrm{Lt}} \frac{\tan\theta}{\theta} = 1.$$

Ex. The arc (if small) of a circle

$$= \frac{8l - \mathrm{L}}{3} \text{ approx.}$$



where l = chord of half the arc,

L = chord of whole arc.

Suppose r = radius, θ angle subtended at centre by arc,

$$\frac{8l - \mathrm{L}}{3} = \frac{8 \cdot 2r\sin\frac{\theta}{4} - 2r\sin\frac{\theta}{2}}{3}$$

$$= \frac{2r}{3}\left(8 \cdot \frac{\theta}{4} - \frac{\theta}{2}\right) \text{ when } \theta \text{ is small,}$$

$$= r\theta = \text{the arc.}$$

200. Limits in sexagesimal measure.

If x'' = the sexagesimal measure of the angle θ radians,

then
$$\frac{x\pi}{180 \times 60 \times 60} = \theta,$$

and
$$\frac{\sin x}{x} = \frac{\sin\theta}{\theta} \cdot \frac{\pi}{180 \times 60 \times 60},$$

and
$$\frac{\tan x}{x} = \frac{\tan\theta}{\theta} \cdot \frac{\pi}{180 \times 60 \times 60}.$$

Hence
$$\underset{x=0}{\mathrm{Lt}} \frac{\sin x}{x} = \mathrm{Lt} \frac{\tan x}{x} = \frac{\pi}{180 \times 60 \times 60}.$$

201. To show
$$\sin\theta > \theta - \frac{\theta^3}{4},$$
$$\cos\theta > 1 - \frac{\theta^2}{2}.$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$= 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}$$
$$= 2\tan\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right)$$
$$= \theta \cdot \frac{\tan\frac{\theta}{2}}{\frac{\theta}{2}}\left\{1 - \frac{\theta^2}{4}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2\right\}.$$

Now by Art. 198,
$$\frac{\tan\frac{\theta}{2}}{\frac{\theta}{2}} > 1 \text{ and } \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} < 1;$$

$$\therefore \sin\theta > \theta\left\{1 - \frac{\theta^2}{4}\right\} > \theta - \frac{\theta^3}{4};$$

again $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$= 1 - 2\cdot\frac{\theta^2}{4}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2$$

$$> 1 - \frac{\theta^2}{2}.$$

202. To show

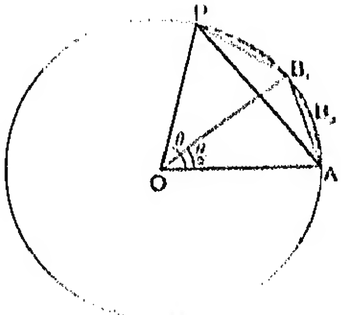
(i) $$\sin\theta > \theta - \frac{\theta^3}{6},$$

(ii) $$\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24},$$

(iii) $$\tan\theta > \theta + \frac{\theta^3}{3} + \frac{\theta^5}{8}.$$

(i) Draw a circle radius r.

Let $\mathrm{A\hat{O}P}$ = the angle θ,
$\mathrm{OB_1}$ bisect AOP,
$\mathrm{OB_2}$ bisect $\mathrm{AOB_1}$,
etc.



Area of sector
$\mathrm{AOP} = \triangle\mathrm{AOP} + \triangle\mathrm{APB_1} + 2\triangle\mathrm{AB_1B_2} + 2^2\triangle\mathrm{AB_2B_3}$ + etc. to infinity.

$$\therefore \tfrac{1}{2}r^2\theta = \tfrac{1}{2}r^2\sin\theta + \tfrac{1}{2}\mathrm{AB_1}^2.\sin\mathrm{P\hat{B}_1A}$$
$$+ 2.\tfrac{1}{2}\mathrm{AB_2}^2.\sin\mathrm{A\hat{B}_2B_1} + 2^2.\tfrac{1}{2}\mathrm{AB_3}^2.\sin\mathrm{A\hat{B}_3B_2} + \ldots\ldots$$
$$= \tfrac{1}{2}r^2\sin\theta + \tfrac{1}{2}\left(2r\sin\frac{\theta}{4}\right)^2.\sin\frac{\theta}{2}$$
$$+ 2.\tfrac{1}{2}\left(2r.\sin\frac{\theta}{8}\right)^2.\sin\frac{\theta}{4} + 2^2.\tfrac{1}{2}\left(2r\sin\frac{\theta}{16}\right)^2.\sin\frac{\theta}{8} + \ldots\ldots$$
$$< \tfrac{1}{2}r^2\sin\theta + \tfrac{1}{2}r^2\left(2.\frac{\theta}{4}\right)^2\frac{\theta}{2} + 2.\frac{r^2}{2}\left(2.\frac{\theta}{2^3}\right)^2.\frac{\theta}{2^2}$$
$$+ 2^2.\frac{r^2}{2}\left(2.\frac{\theta}{2^4}\right)^2.\frac{\theta}{2^3} + \ldots\ldots;$$

$$\therefore \quad \theta < \sin\theta + \frac{\theta^3}{8}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\ldots\right)$$

$$< \sin\theta + \frac{\theta^3}{8}\cdot\frac{1}{1-\frac{1}{4}}$$

$$< \sin\theta + \frac{\theta^3}{8}\cdot\frac{4}{3};$$

$$\therefore \quad \sin\theta > \theta - \frac{\theta^3}{6}.$$

(ii) $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$< 1 - 2\left\{\frac{\theta}{2} - \frac{\left(\frac{\theta}{2}\right)^3}{6}\right\}^2$$

$$< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{1152}.$$

Hence $$\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

Lemma (A) $$1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} \text{ is positive,}$$

$$\cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

and $\cos\theta$ is positive since θ is a positive acute angle.

$$\therefore \ 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} \text{ is positive.}$$

Lemma (B) $$\frac{7\theta^7}{144} - \frac{\theta^9}{192} \text{ is positive,}$$

$$\frac{7\theta^7}{144} - \frac{\theta^9}{192} = \frac{\theta^7}{192}\left\{\frac{7\times 192}{144} - \theta^2\right\}$$

$$= \frac{\theta^7}{192}\cdot\frac{1}{4}\cdot\{37\tfrac{1}{3} - (2\theta)^2\}.$$

Now
$$(2\theta)^2 < \pi^2$$
$$< 16$$
$$< 37\tfrac{1}{3};$$
$$\therefore \frac{7\theta^7}{144} - \frac{\theta^9}{192} \text{ is positive.}$$

(iii)
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$> \frac{\theta - \dfrac{\theta^3}{6}}{1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}}$$
$$> \theta + \frac{\theta^3}{3} + \frac{\theta^5}{8} + \frac{\dfrac{7\theta^7}{144} - \dfrac{\theta^9}{192}}{1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}}, \text{ by division}$$
$$> \theta + \frac{\theta^3}{3} + \frac{\theta^5}{8}.$$

203. To show that $\dfrac{\sin\theta}{\theta}$ continually decreases as θ increases from 0 to $\dfrac{\pi}{2}$.

We have only to show
$$\frac{\sin\theta}{\theta} - \frac{\sin(\theta+\alpha)}{\theta+\alpha}$$
is positive when α is acute.

$$\text{Expression} = \frac{(\theta+\alpha)\sin\theta - \theta\,.\sin(\theta+\alpha)}{\theta(\theta+\alpha)}$$
$$= \frac{\theta\sin\theta(1-\cos\alpha) + (\alpha\sin\theta - \theta\cos\theta\sin\alpha)}{\theta(\theta+\alpha)}$$
$$= \text{a positive quantity} + \frac{\alpha\sin\theta - \theta\cos\theta\sin\alpha}{\theta(\theta+\alpha)}$$

$$= \text{a positive quantity} + \frac{\dfrac{\tan\theta}{\theta} - \dfrac{\sin\alpha}{\alpha}}{\dfrac{\theta(\theta+\alpha)}{\theta\,\alpha\cos\theta}}$$

$$= \text{a positive quantity},$$

so $$\frac{\tan\theta}{\theta} > 1 \quad \text{and} \quad \frac{\sin\alpha}{\alpha} < 1.$$

similar way $\dfrac{\tan\theta}{\theta}$ continually increases.

04. Euler's Theorem.

$$\mathop{\mathrm{Lt}}_{n=\infty} \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ \ldots\ldots\ \cos\frac{\theta}{2^n} \ \text{ is } \ \frac{\sin\theta}{\theta}.$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}$$

$$= 2^3\sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}, \text{ etc.}$$

$$\therefore\ \frac{\sin\theta}{2^n\sin\dfrac{\theta}{2^n}} = \cos\frac{\theta}{2^n}\cos\frac{\theta}{2^{n-1}}\ \ldots\ldots\ \cos\frac{\theta}{2^2}\cos\frac{\theta}{2};$$

when n is indefinitely increased

$$\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ \ldots\ldots\ \cos\frac{\theta}{2^n} = \mathop{\mathrm{Lt}}_{n=\infty} \frac{\sin\theta}{\theta\,\dfrac{\sin\dfrac{\theta}{2^n}}{\dfrac{\theta}{2^n}}}$$

$$= \frac{\sin\theta}{\theta}.$$

205. Ex. 1. Find the values of $\sin 8'$ and $\cos 8'$ $[\pi = 3{\cdot}14159]$.

$$8' = \frac{8\pi}{180 \times 60} \text{ radians} = \frac{8 \times 3{\cdot}14159}{180 \times 60} \text{ (approx.)}$$
$$= {\cdot}0023271 \text{ radians};$$

therefore, since $\quad \sin\theta = \theta - \frac{\theta^3}{4}$ (approx.),

$$\sin 8' = {\cdot}0023271 - \frac{({\cdot}0023271)^3}{4}$$
$$= {\cdot}0023271 - {\cdot}000000002\ldots$$
$$= {\cdot}0023271 \text{ (nearly)}.$$

Also $\quad \cos\theta = 1 - \frac{\theta^2}{2}$ (approx.)

$$= 1 - \tfrac{1}{2}({\cdot}0023271)^2$$
$$= 1 - {\cdot}000002707$$
$$= {\cdot}9999973.$$

Ex. 2. If $\dfrac{\sin\theta}{\theta} = \dfrac{483}{484}$ find an approximate value for θ.

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{4}}{\theta} = 1 - \frac{\theta^2}{4};$$

$$\therefore\ 1 - \frac{\theta^2}{4} = \frac{483}{484},$$

or $\quad \theta^2 = \frac{1}{121};$

$$\therefore\ \theta = \tfrac{1}{11} \text{ radian}.$$

Ex. 3. Solve approximately $\sin\left(\dfrac{\pi}{3} + \theta\right) = {\cdot}87$.

Expanding $\quad \dfrac{\sqrt{3}}{2}\cos\theta + \dfrac{1}{2}\sin\theta = {\cdot}87,$

and since

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = {\cdot}866,$$

θ is a small angle,

$$\therefore\ \frac{\sqrt{3}}{2} + \frac{1}{2}\theta = {\cdot}87;$$

$$\therefore\ \theta = 1{\cdot}74 - 1{\cdot}73205$$
$$= {\cdot}00795 \text{ radian}.$$

Ex. 4. Find the angle subtended by a kilometre-stone 1 metre high at a place 1·5 kilometres off $[\pi = \frac{22}{7}]$.

If θ is the number of radians in the angle

$$\theta = \tan\theta = \frac{1 \text{ metre}}{1\cdot5 \text{ kilom.}}$$

$$= \frac{1}{1500} \text{ radians}$$

$$= \frac{180 \times 7 \times 60'}{22 \times 1500} = \frac{126'}{55}$$

$$= 2'\cdot29.$$

EXAMPLES XLIV.

1. Find the values of $\sin 5'$ and $\cos 5'$ $[\pi = 3\cdot14159]$.

2. If $\dfrac{\sin\theta}{\theta} = \dfrac{675}{676}$, find the value of θ.

3. Find the values of $\sin 3'$ and $\cos 3'$ $[\pi = 3\cdot14159]$.

4. Solve the equation $\sin\left(\dfrac{\pi}{4} + \theta\right) = \cdot71$.

5. Calculate the approximate value of θ, when $\cos\theta = \frac{1681}{1682}$.

6. If $\sin\theta = \frac{1145}{1186}\theta$, find the value of θ.

7. Find the value of θ from the equation $\cos\left(\dfrac{\pi}{3} - \theta\right) = \cdot51$.

8. Find θ when $\cos\theta = \frac{2737}{2738}$.

9. Solve the equation $\tan\left(\dfrac{\pi}{3} + \theta\right) = 1\cdot73$.

10. If $\dfrac{\sin\theta}{\theta} = \dfrac{1763}{1764}$, calculate the approximate value of θ.

11. A post, 1 foot high, stands at the top of a tower of height 150 feet; calculate (to $\frac{1}{100}$ of a minute) the angle it subtends at a point on the ground 900 feet from the foot of the tower. $[\pi = \frac{22}{7}.]$

12. Find the value of $\cos\left(\dfrac{\pi}{6} + \theta\right)$, when $\theta = \cdot005$ radian.

13. An object, 880 metres off, subtends an angle of 51′ at the observer's eye: find the length of the object. $(\pi = \frac{22}{7}.)$ [Answer to 1 centimetre.]

14. A cliff 180 metres high is surmounted by a flagstaff which subtends an angle of ·035 radian at a point on the ground 350 metres from the foot of the cliff. Find the height of the flagstaff to the nearest decimetre.

CHAPTER XXI.

SUMMATION OF SERIES.

206. To find the sum of the sines of n series of angles in A.P.

Let $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$
$$+ \sin\{\alpha+(n-1)\beta\} = S;$$

$$\therefore\ 2S\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + \ldots\ldots$$
$$+ 2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}.$$

Now
$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right)$$
$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right)$$
$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right)$$

....

$$2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{2n-3}{2}\beta\right)$$
$$- \cos\left(\alpha+\frac{2n-1}{2}\beta\right);$$

y addition

$$\sin\frac{\beta}{2}=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$=2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2};$$

$$\therefore\ S=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

To find the sum of the cosines of a series in A.P.

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots$$
$$+\cos\{\alpha+(n-1)\beta\}=C.$$

ne of this series may either be deduced from the ing $\alpha=\frac{\pi}{2}+\alpha$, whence we obtain

$$C=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}},$$

worked out independently.

$$\frac{\beta}{2}=2\cos\alpha\sin\frac{\beta}{2}+2\cos(\alpha+\beta)\sin\frac{\beta}{2}+\ldots\ldots$$
$$+2\cos\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}.$$

$$2\cos\alpha\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right).$$

$$s\,(\alpha+\beta)\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)$$

$$2\cos(\alpha+2\beta)\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{5\beta}{2}\right)-\sin\left(\alpha+\frac{3\beta}{2}\right)$$

....

$$2\cos\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)$$
$$-\sin\left(\alpha+\frac{2n-3}{2}\beta\right);$$

therefore, by addition

$$2\mathrm{C}\sin\frac{\beta}{2}=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$
$$=2\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2};$$

$$\therefore\ \mathrm{C}=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

208. It should be observed that

$$\alpha+\frac{n-1}{2}\beta=\tfrac{1}{2}\left[\alpha+\{\alpha+(n-1)\beta\}\right]$$
$$=\tfrac{1}{2}\,(\text{sum of first and last angle}),$$

and that the two results only differ in the first term of the numerator.

209. $\sin\frac{n\beta}{2}=0$, when $\frac{n\beta}{2}=k\pi$ or $\beta=\frac{2k\pi}{n}$,

k being an integer; and in this case both S and C vanish.

Thus the sum of the sines or cosines of n angles in A.P. vanishes when the common difference of the angle, β, is a multiple of $\frac{2\pi}{n}$.

212. Several other series can be summed by decomposing each term into the difference of two others.

Ex. 3. Find the value of

$$\sin\alpha\cos 5\alpha+\sin 3\alpha\cos 7\alpha+\sin 5\alpha\cos 9\alpha+\ldots \text{ to } n \text{ terms.}$$

$$\begin{aligned}2S&=(\sin 6\alpha-\sin 4\alpha)+(\sin 10\alpha-\sin 4\alpha)+(\sin 14\alpha-\sin 4\alpha)+\ldots\\&=(\sin 6\alpha+\sin 10\alpha+\sin 14\alpha+\ldots)\\&\quad-(\sin 4\alpha+\sin 4\alpha+\sin 4\alpha+\ldots)\\&=\frac{\sin(2n\alpha+4\alpha)\sin 2n\alpha}{\sin 2\alpha}-n\sin 4\alpha.\end{aligned}$$

Ex. 4. Find the value of

$$\frac{1}{\sin\alpha\sin 3\alpha}+\frac{1}{\sin 3\alpha\sin 5\alpha}+\frac{1}{\sin 5\alpha\sin 7\alpha}+\ldots.$$

Since $$\frac{\sin 2\alpha}{\sin\alpha\sin 3\alpha}=\frac{\sin(3\alpha-\alpha)}{\sin\alpha\sin 3\alpha}=\cot\alpha-\cot 3\alpha$$

$$\frac{\sin 2\alpha}{\sin 3\alpha\sin 5\alpha}=\frac{\sin(5\alpha-3\alpha)}{\sin 3\alpha\sin 5\alpha}=\cot 3\alpha-\cot 5\alpha$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\sin 2\alpha}{\sin(2n-1)\alpha\sin(2n+1)\alpha}=\frac{\sin(\overline{2n+1}\,\alpha-\overline{2n-1}\,\alpha)}{\sin(2n-1)\alpha\sin(2n+1)\alpha}$$
$$=\cot(2n-1)\alpha-\cot(2n+1)\alpha;$$

therefore, adding
$$\sin 2\alpha\,.\,S=\cot\alpha-\cot(2n+1)\alpha$$

or
$$S=\frac{\cot\alpha-\cot(2n+1)\alpha}{\sin 2\alpha}.$$

Ex. 5. Find the value of

$$\operatorname{cosec}\alpha+\operatorname{cosec}2\alpha+\operatorname{cosec}4\alpha+\ldots \text{ to } n \text{ terms.}$$

Since $$\operatorname{cosec}\alpha=\cot\frac{\alpha}{2}-\cot\alpha$$

$$\operatorname{cosec}2\alpha=\cot\alpha-\cot 2\alpha$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\operatorname{cosec}2^{n-1}\alpha=\cot 2^{n-2}\alpha-\cot 2^{n-1}\alpha;$$

therefore, adding $$S=\cot\frac{\alpha}{2}-\cot 2^{n-1}\alpha.$$

EXAMPLES XLV.

Sum the following series to n terms:

1. $\sin 2A + \sin 5A + \sin 8A + \ldots$

2. $\cos A + \cos 3A + \cos 5A + \ldots$

3. $\cos \frac{A}{3} + \cos \frac{4A}{3} + \cos \frac{7A}{3} + \ldots$

4. $\cos \theta + \cos\left(\theta + \frac{\pi}{n}\right) + \cos\left(\theta + \frac{2\pi}{n}\right) + \ldots$

5. $\sin \alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \ldots$

6. $\sin \frac{A}{2} + \sin A + \sin \frac{3A}{2} + \ldots$

Find the sum of:

7. $\sin \frac{\pi}{21} + \sin \frac{3\pi}{21} + \sin \frac{5\pi}{21} + \ldots + \sin \frac{19\pi}{21}$.

8. $\cos \frac{\pi}{23} + \cos \frac{3\pi}{23} + \cos \frac{5\pi}{23} + \ldots + \cos \frac{21\pi}{23}$.

9. $\sin \frac{\pi}{2n-1} + \sin \frac{3\pi}{2n-1} + \sin \frac{5\pi}{2n-1} + \ldots$ to n terms.

Find the sum to n terms of:

10. $\sin \alpha - \sin 2\alpha + \sin 3\alpha - \ldots$

11. $\cos 2\alpha - \cos 4\alpha + \cos 6\alpha - \ldots$

12. $\sin 2\alpha - \sin\left(2\alpha + \frac{\pi}{n}\right) + \sin\left(2\alpha + \frac{2\pi}{n}\right) - \ldots$

13. $\cos 3\alpha - \cos\left(3\alpha - \frac{\pi}{n}\right) + \cos\left(3\alpha - \frac{2\pi}{n}\right) - \ldots$

14. $\sin \alpha \sin 3\alpha + \sin 3\alpha \sin 5\alpha + \sin 5\alpha \sin 7\alpha + \ldots$

15. $\cos \alpha \cos 3\alpha + \cos 3\alpha \cos 5\alpha + \cos 5\alpha \cos 7\alpha + \ldots$

16. $\sin \theta \cos 4\theta + \sin 3\theta \cos 6\theta + \sin 5\theta \cos 8\theta + \ldots$

17. $\frac{1}{\cos \alpha \cos 3\alpha} + \frac{1}{\cos 3\alpha \cos 5\alpha} + \frac{1}{\cos 5\alpha \cos 7\alpha} + \ldots$

18. $\dfrac{1}{\sin\alpha\sin 4\alpha}+\dfrac{1}{\sin 4\alpha\sin 7\alpha}+\dfrac{1}{\sin 7\alpha\sin 10\alpha}+\dots$

19. $\sec 2\alpha\sec 4\alpha+\sec 4\alpha\sec 6\alpha+\sec 6\alpha\sec 8\alpha+\dots$

20. $\operatorname{cosec} 2\alpha\operatorname{cosec} 3\alpha+\operatorname{cosec} 3\alpha\operatorname{cosec} 4\alpha+\operatorname{cosec} 4\alpha\operatorname{cosec} 5\alpha+\dots$

21. $\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\dots$

22. $\cos^2 2\alpha+\cos^2 3\alpha+\cos^2 4\alpha+\dots$

23. $\sin^2\alpha+\sin^2\left(\alpha+\dfrac{\pi}{n}\right)+\sin^2\left(\alpha+\dfrac{2\pi}{n}\right)+\dots$

24. $\cos^3\alpha+\cos^3(\alpha+\beta)+\cos^3(\alpha+2\beta)+\dots$

25. $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\dots$

26. $\sin^4\alpha+\sin^4 2\alpha+\sin^4 3\alpha+\dots$

27. $\cos^4\alpha+\cos^4 3\alpha+\cos^4 5\alpha+\dots$

Find the sum to n terms of:

28. $\sin\alpha+\sin\dfrac{n-4}{n-2}\alpha+\sin\dfrac{n-6}{n-2}\alpha+\dots$

29. $\dfrac{1}{\cos\theta+\cos 3\theta}+\dfrac{1}{\cos\theta+\cos 5\theta}+\dfrac{1}{\cos\theta+\cos 7\theta}+\dots$

30. $\sin\alpha-\sin 2\alpha+\sin 3\alpha-\sin 4\alpha+\dots$

31. $\cos\alpha\cos 2\alpha\cos 3\alpha+\cos 2\alpha\cos 3\alpha\cos 4\alpha+\dots$

32. $\tan^{-1}\dfrac{x}{1+1\,.\,2x^2}+\tan^{-1}\dfrac{x}{1+2\,.\,3x^2}+\dots$

33. $\tan^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}+\tan^{-1}\dfrac{1}{1+3+3^2}+\dots$

Prove that:

34. $\dfrac{\sin\alpha+\sin 2\alpha+\sin 3\alpha+\dots+\sin n\alpha}{\cos\alpha+\cos 2\alpha+\cos 3\alpha+\dots+\cos n\alpha}=\tan\dfrac{n+1}{2}\alpha.$

35. If $A_1, A_2\dots A_{2n+1}$ are the angular points of a regular polygon inscribed in a circle and O a point on the arc between A_1 and A_{2n+1}; prove that

$$OA_1+OA_3+\dots+OA_{2n+1}=OA_2+OA_4+\dots+OA_{2n}.$$

36. From any point on the circumference of a circle of radius r, chords are drawn to the angular points of the regular inscribed polygon of n sides. Show that the sum of the squares of the chords is $2nr^2$.

CHAPTER XXII.

EXPONENTIAL THEOREM.

213. It is proved in Algebra that

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$

is a one-valued, continuous, convergent series for all real values of x; we shall denote it by $E(x)$.

214. To prove

$$E(x) \times E(y) = E(x + y).$$

The general term of

$$E(x) \times E(y)$$

$$= \frac{x^r}{\lfloor r} + \frac{x^{r-1}}{\lfloor r-1} \cdot \frac{y}{\lfloor 1} + \frac{x^{r-2}}{\lfloor r-2} \cdot \frac{y^2}{\lfloor 2} + \ldots\ldots + \frac{y^r}{\lfloor r}$$

$$= \frac{1}{\lfloor r}\left[x^r + rx^{r-1}y + \frac{r(r-1)}{\lfloor 2}x^{r-2}y^2 + \ldots\ldots + y^r\right]$$

$= \dfrac{(x+y)^r}{\lfloor r}$ assuming the Binomial Theorem for a positive integral index

$=$ general term of $E(x + y)$;

$$\therefore\ E(x) \times E(y) = E(x + y).$$

Similarly

$$E(x) \times E(y) \times E(z) \ldots\ldots = E(x + y + z + \ldots\ldots).$$

215. To prove $\{\mathsf{E}(1)\}^x = \mathsf{E}(x)$.

1st when x is a positive integer.

By Art. 214

$$\mathsf{E}(1) \times \mathsf{E}(1) \times \ldots\ldots \text{ to } x \text{ factors}$$
$$= \mathsf{E}(1 + 1 + 1 + \ldots\ldots \text{ to } x \text{ terms})$$
$$= \mathsf{E}(x);$$
$$\therefore \{\mathsf{E}(1)\}^x = \mathsf{E}(x).$$

2nd when x is a positive fraction $= \dfrac{h}{k}$.

By Art. 214

$$\left\{\mathsf{E}\left(\frac{h}{k}\right)\right\}^k = \mathsf{E}\left(\frac{h}{k} + \frac{h}{k} + \ldots\ldots \text{ to } k \text{ terms}\right)$$
$$= \mathsf{E}(h) = \{\mathsf{E}(1)\}^h;$$
$$\therefore \mathsf{E}\left(\frac{h}{k}\right) = \{\mathsf{E}(1)\}^{\frac{h}{k}};$$
$$\therefore \mathsf{E}(x) = \{\mathsf{E}(1)\}^x.$$

3rd when x is negative $= -h$.

Then by Art. 214

$$\mathsf{E}(-h) \times \mathsf{E}(h) = \mathsf{E}(0) = 1;$$
$$\therefore \mathsf{E}(-h) = \frac{1}{\mathsf{E}(h)};$$
$$\therefore \mathsf{E}(x) = \frac{1}{\mathsf{E}(h)} = \frac{1}{\{\mathsf{E}(1)\}^h} = \{\mathsf{E}(1)\}^{-h}$$
$$= \{\mathsf{E}(1)\}^x.$$

216. $\mathsf{E}(1) = 1 + \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \ldots\ldots$

is generally denoted by e.

Thus $e^x = \{\mathsf{E}(1)\}^x = \mathsf{E}(x) = 1 + \dfrac{x}{\lfloor 1} + \dfrac{x^2}{\lfloor 2} + \dfrac{x^3}{\lfloor 3} + \ldots\ldots$

This is called the Exponential Theorem and it has been proved for any *real* commensurable exponent.

217. *To prove that e is incommensurable.*

Suppose it is commensurable and equal to $\frac{m}{n}$, m and n being integers, then

$$\frac{m}{n} = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots\ldots \frac{1}{\underline{|n}} + \frac{1}{\underline{|n+1}} + \ldots\ldots;$$

multiplying by $\underline{|n}$

$$m\underline{|n-1} = \text{a whole number} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots$$

But
$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots$$
$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots\ldots$$
$$< \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}}$$
$$< \frac{1}{n};$$

$$\therefore \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots \neq \text{a whole number},$$

thus $m\underline{|n-1} =$ a whole number + a fraction, which is impossible.

$\therefore$ e is incommensurable.

218. Logarithmic series.

$$a^n = e^{\log_e a^n} = e^{n \log_e a}$$
$$= 1 + (n \log_e a) + \frac{(n \log_e a)^2}{\underline{|2}} + \frac{(n \log_e a)^3}{\underline{|3}} + \ldots\ldots$$

Let $a = 1 + x$, x being a proper fraction, positive or negative, then

$$(1+x)^n = 1 + n \log_e (1+x) + \frac{\{n \log_e (1+x)\}^2}{\underline{|2}} + \ldots\ldots;$$

$$\therefore\ 1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}x^3+\dots\dots$$

$$=1+n\log_e(1+x)+\frac{\{n\log_e(1+x)\}^2}{\lfloor 2}+\dots\dots.$$

Both series are convergent and therefore we may equate the coefficients of n;

$$\therefore\ \log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\dots$$

Changing x into $-x$, we have

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots\dots$$

These series are convergent when x is limited as above;

$$\therefore\ \log_e\frac{1+x}{1-x}=\log_e(1+x)-\log_e(1-x)$$

$$=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\dots\right).$$

219. Calculation of Logarithms.

In the above series put $\dfrac{m}{n}=\dfrac{1+x}{1-x}$, then

$$\log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\dots\dots\right\}.$$

Put $m=2,\ n=1.$

$$\log_e 2=2\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\dots\dots\right\}$$

$$=2\left\{\begin{array}{r}\cdot333333333\\ \cdot012345679\\ 823045\\ 65324\\ 5645\\ 513\\ 48\\ \hline \cdot3465736\end{array}\right.$$

$=\cdot693147$ (correct to six places).

Also, by putting $m=3$, $n=2$,

$$\log_e 3 - \log_e 2 = 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \dots\dots\right\}$$

$$= \cdot405465;$$

$$\therefore \log_e 3 = 1\cdot09861 \text{ (correct to five places)}$$

and so on.

220. *To find the limiting values of*

$$\left(\cos\frac{\alpha}{n}\right)^n \text{ and } \left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n$$

when n is indefinitely increased.

Let $\quad x = \left(\cos\frac{\alpha}{n}\right)^n = \left(1 - \sin^2\frac{\alpha}{n}\right)^{\frac{n}{2}}$;

then $\quad \log_e x = \frac{n}{2}\log_e\left(1 - \sin^2\frac{\alpha}{n}\right)$

$$= -\frac{n}{2}\left(\sin^2\frac{\alpha}{n} + \frac{1}{2}\sin^4\frac{\alpha}{n} + \dots\dots\right)$$

$$= -\frac{n}{2}\sin\frac{\alpha}{n}\left(\sin\frac{\alpha}{n} + \frac{1}{2}\sin^3\frac{\alpha}{n} + \dots\dots\right);$$

now $\quad \mathop{\mathrm{Lt}}_{n=\infty}\frac{n}{2}\sin\frac{\alpha}{n} = \frac{\alpha}{2}$ (Art. 199),

and $\quad \mathop{\mathrm{Lt}}_{n=\infty}\left(\sin\frac{\alpha}{n} + \frac{1}{2}\sin^3\frac{\alpha}{n} + \dots\dots\right) = 0.$

Therefore in the limit $\log_e x = 0$, therefore $x = 1$;

$$\therefore \mathop{\mathrm{Lt}}_{n=\infty}\left(\cos\frac{\alpha}{n}\right)^n = 1.$$

Now $\quad 1 > \dfrac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}} > \dfrac{\sin\frac{\alpha}{n}}{\tan\frac{\alpha}{n}}\left(\text{or } \cos\frac{\alpha}{n}\right)$ (Art. 198);

9. From identity $\left(\dfrac{x+y}{x-y}\right)^2 = \dfrac{1+\dfrac{2xy}{x^2+y^2}}{1-\dfrac{2xy}{x^2+y^2}}$,

prove that $\quad \log_e \dfrac{x+y}{x-y} = \dfrac{2xy}{x^2+y^2} + \dfrac{1}{3}\left(\dfrac{2xy}{x^2+y^2}\right)^3 + \dfrac{1}{5}\left(\dfrac{2xy}{x^2+y^2}\right)^5 + \dots$

10. Prove that

$$\log_e(x+1) - \log_e x = 2\left\{\frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \dots\right\},$$

and deduce that

$$\log_e 13 = 2\log_e 2 + \log_e 3 + \cdot 0800427\dots$$

11. $\log_e \operatorname{cosec}\theta = \frac{1}{2}\cos^2\theta + \frac{1}{4}\cos^4\theta + \frac{1}{6}\cos^6\theta + \dots$

12. $\log_e \operatorname{cosec}\theta = \frac{1}{2}\cot^2\theta - \frac{1}{4}\cot^4\theta + \frac{1}{6}\cot^6\theta - \dots$

13. $\frac{1}{2}\log_e \dfrac{\sin\left(\theta+\frac{\pi}{4}\right)}{\sin\left(\theta-\frac{\pi}{4}\right)} = \cot\theta + \frac{1}{3}\cot^3\theta + \frac{1}{5}\cot^5\theta + \dots$

14. $\sin\theta + \frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + \dots$

$$= 2\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2} + \frac{1}{5}\tan^5\frac{\theta}{2} + \dots\right),$$

where $$\theta > 0 < \frac{\pi}{2};$$

use $$\frac{1+\sin\theta}{1-\sin\theta} = \left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)^2.$$

15. $\log_e \sin 2\theta - \log_e \tan\theta = \cos 2\theta - \frac{1}{2}\cos^2 2\theta + \frac{1}{3}\cos^3 2\theta - \dots$

16. If $a = \cdot 9999999999$ and $e = 2\cdot 71828$,

prove that $\quad a + \frac{1}{2}a^2 + \frac{1}{3}a^3 + \dots = 23\cdot 02$ approx.

17. Prove that the coefficient of x^n in the expansion of

$$\log_e(1+x+x^2+\ldots x^{m-1})$$

is either

$$-\frac{m-1}{n} \quad \text{or} \quad \frac{1}{n},$$

according as n is, or is not, a multiple of m.

18. Prove that the limit of $\left(\cos\dfrac{a}{n}\right)^{n^x}$ when x is an integer and n indefinitely increased is

$$\begin{aligned} &0 \text{ when } x > 2, \\ &e^{-\frac{a^2}{2}} \quad ,, \quad x = 2, \\ &1 \quad ,, \quad x < 2. \end{aligned}$$

CHAPTER XXIII.

DE MOIVRE'S THEOREM.

221. In this chapter i stands for $\sqrt{-1}$.

Thus $\qquad i^2 = -1; \quad i^3 = -i; \quad i^4 = 1; \quad$ etc.

when
$$a + ib = a' + ib',$$
a, b, a', b' being real; it is assumed
$$a = a'; \quad b = b'.$$

Such an expression as $a + ib$ is called a complex quantity.

222. De Moivre's theorem. To show that

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, when n is integral,

and that one of the values of

$(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$, when n is fractional.

(i) *When n is a positive integer.*

By actual multiplication

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$
$$= \cos(\alpha + \beta) + i\sin(\alpha + \beta);$$

$$\therefore\ (\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$$
$$=\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}(\cos\gamma+i\sin\gamma)$$
$$=\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)$$

and so on.

Putting $\alpha=\beta=\gamma=\ldots\ldots=\theta,$

and supposing there are n letters α, β, γ ...... we have

$$(\cos\boldsymbol{\theta}+i\sin\boldsymbol{\theta})^n=\cos n\boldsymbol{\theta}+i\sin n\boldsymbol{\theta}.$$

Thus the theorem is established for a positive integer.

(ii) *When n is a negative integer $=-m$ suppose*

$$(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m}=\frac{1}{(\cos\theta+i\sin\theta)^m}$$
$$=\frac{1}{\cos m\theta+i\sin m\theta}=\frac{\cos m\theta-i\sin m\theta}{\cos^2 m\theta+\sin^2 m\theta}$$
$$=\cos(-m)\theta+i\sin(-m)\theta$$
$$=\cos n\theta+i\sin n\theta.$$

Thus the theorem is established for *any* integer.

(iii) *When n is any fraction $=\frac{h}{k}$* suppose, h, k being integers.

By (i) and (ii) $\left(\cos\frac{\theta}{k}+i\sin\frac{\theta}{k}\right)^k=\cos\theta+i\sin\theta$ when k is integral;

$\therefore\ \left(\cos\frac{\theta}{k}+i\sin\frac{\theta}{k}\right)$ is one of the values of $(\cos\theta+i\sin\theta)^{\frac{1}{k}}$;

$\therefore\ \left(\cos\frac{\theta}{k}+i\sin\frac{\theta}{k}\right)^h$ „ „ „ $(\cos\theta+i\sin\theta)^{\frac{h}{k}}$;

i.e. by (i) and (ii),

$\cos\frac{h}{k}\theta+i\sin\frac{h}{k}\theta$ „ „ „ $(\cos\theta+i\sin\theta)^{\frac{h}{k}}$;

i.e. $\cos n\theta+i\sin n\theta$ „ „ „ $(\cos\theta+i\sin\theta)^n$.

Thus the theorem is established for any commensurable number.

223. We have just shown that

$\cos\frac{h}{k}\theta + i\sin\frac{h}{k}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{h}{k}}$;

we now find the others.

Since $(\cos\theta + i\sin\theta)^{\frac{h}{k}} = (\cos h\theta + i\sin h\theta)^{\frac{1}{k}}$,

it follows that we have merely to find the other values of

$$(\cos h\theta + i\sin h\theta)^{\frac{1}{k}}.$$

Putting $h\theta + 2m\pi$ for $h\theta$, where $m =$ a positive integer, we have

$$\cos\left(\frac{h\theta}{k} + \frac{2m\pi}{k}\right) + i\sin\left(\frac{h\theta}{k} + \frac{2m\pi}{k}\right)$$

$$= \text{one of the values of}$$

$$\{\cos(h\theta + 2m\pi) + i\sin(h\theta + 2m\pi)\}^{\frac{1}{k}}$$

$$= \text{one of the values of}$$

$$(\cos h\theta + i\sin h\theta)^{\frac{1}{k}}.$$

Hence by putting

$$m = 0, 1, 2, 3, \ldots\ldots, (k-1),$$

we obtain k values of

$$(\cos h\theta + i\sin h\theta)^{\frac{1}{k}}$$

and these values are all different; for suppose any two are equal,

$$\cos\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right) + i\sin\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right)$$

$$= \cos\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right) + i\sin\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right).$$

Then equating the real and imaginary parts

$$\cos\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right) = \cos\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right)$$

and $$\sin\left(\frac{h\theta}{k} + \frac{2r\pi}{k}\right) = \sin\left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right);$$

Ex. 2. Extract the cube roots of unity.

Here $a=1;\quad b=0;\quad r=1;\quad \theta=0.$

Hence the roots are

$$\cos 0+i\sin 0;\quad i.e.\ 1.$$

$$\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3};\quad i.e.\ \frac{-1+i\sqrt{3}}{2}.$$

$$\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3};\quad i.e.\ \frac{-1-i\sqrt{3}}{2}.$$

Ex. 3. If $x+\frac{1}{x}=2\cos A$, prove that $x^n+\frac{1}{x^n}=2\cos nA.$

Since $$x+\frac{1}{x}=2\cos A;$$

$$\therefore\ x^2-2x\cos A+1=0.$$

Solving this quadratic in x,

$$x=\cos A\pm i\sin A.$$

Taking the positive sign

$$x^n=(\cos A+i\sin A)^n=\cos nA+i\sin nA,$$

$$x^{-n}=(\cos A+i\sin A)^{-n}=\cos nA-i\sin nA,$$

$$\therefore\ x^n+x^{-n}=2\cos nA.$$

Similarly for the negative sign.

Ex. 4. Find the value of

$$\frac{(\cos 5\theta+i\sin 5\theta)^2}{(\cos 3\theta-i\sin 3\theta)^3}.$$

$$\text{Exp}^{\text{n}}.=\frac{(\cos\theta+i\sin\theta)^{10}}{(\cos\theta+i\sin\theta)^{-9}}=(\cos\theta+i\sin\theta)^{19}$$

$$=\cos 19\theta+i\sin 19\theta.$$

Ex. 5. If

$$\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0,$$

prove that

$$\cos 4\alpha+\cos 4\beta+\cos 4\gamma$$
$$=2\{\cos 2(\beta+\gamma)+\cos 2(\gamma+\alpha)+\cos 2(\alpha+\beta)\}.$$

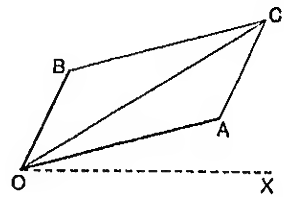
Let

$$\cos\alpha+i\sin\alpha=a,\quad \cos\beta+i\sin\beta=b,\quad \cos\gamma+i\sin\gamma=c.$$

Then $$a+b+c=0.$$

226. The addition of vectors or complex quantities.

Let $\overline{OA}$ and $\overline{OB}$ be two vectors, complete the parallelogram OACB. Then by Art. 225



$$\overline{OA}+\overline{OB}\equiv\overline{OA}+\overline{AC}$$

= the carrying of a tracing point from O to A and then to C

$$=\overline{OC};$$

also
$$\overline{OB}+\overline{OA}\equiv\overline{OB}+\overline{BC}$$
$$=\overline{OC},$$
$$\therefore\ \overline{OA}+\overline{OB}=\overline{OB}+\overline{OA}.$$

Thus vectors or complex quantities when added obey the Commutative Law and the sum of any two is represented by the diagonal of a parallelogram having the two as adjacent sides.

By making $A\hat{O}X=B\hat{O}X=n\pi$ (n being any integer) AO and BO become collinear and we obtain the sum of two numbers as arithmetically defined.

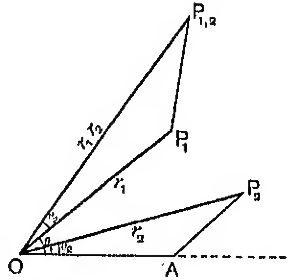
227. The multiplication of complex quantities.

To multiply a by b, we do to a what must be done to unity to obtain b.

Let (r_1, θ_1), (r_2, θ_2) be two complex quantities.

To obtain (r_2, θ_2) from unity we multiply the unit by r_2 and revolve the resulting length through the angle θ_2. [See triangle AOP_2.]

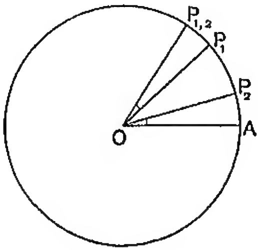
Hence to multiply (r_1, θ_1) by (r_2, θ_2), multiply r_1 by r_2 and thus obtain r_1r_2 for new modulus and then rotate this length from the position θ_1 through an angle θ_2. [See triangle $P_1OP_{1,2}$.] The triangles AOP_2 and $P_1OP_{1,2}$ are seen to be similar;



thus
$$(r_1, \theta_1)\times(r_2, \theta_2)=(r_1r_2, \theta_1+\theta_2).$$

229. If in Art. 228 and Fig. Art. 227 we put r_1 and r_2 both equal to unity, we have a geometrical representation of

$$(\cos\theta_1 + i\sin\theta_1) \times (\cos\theta_2 + i\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2),$$



and we see that to multiply one complex quantity by any other, the modulus being unity in both cases, we have merely to rotate the line representing (1, 0) through an angle equal to the sum of the amplitudes of the two quantities; and thus in general, to multiply together any number of complex quantities which have a common modulus unity, we have merely to rotate the line (1, 0) through an angle equal to the sum of the amplitudes of the quantities. And if all the amplitudes are equal we have a geometrical representation of

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

i.e. of De Moivre's theorem for a positive index. We thus see also that one of the values of

$$(\cos n\theta + i\sin n\theta)^{\frac{1}{n}}$$

is obtained by rotating the line (1, 0) through an angle $\frac{1}{n}$th of the amplitude of the line whose nth root is indicated.

We can now show how to geometrically represent the other values of

$$(\cos\phi + i\sin\phi)^{\frac{1}{n}}.$$

(iv) when OP is at OA the 4th time, OR is at OR_4, $XOR_4 = \dfrac{6\pi+\phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{6\pi+\phi}{5} + i\sin\frac{6\pi+\phi}{5}\right);$$

(v) when OP is at OA the 5th time, OR is at OR_5, $XOR_5 = \dfrac{8\pi+\phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{8\pi+\phi}{5} + i\sin\frac{8\pi+\phi}{5}\right);$$

when OP is at OA the 6th time, OR is at OR_1, the 2nd time;

when OP is at OA the 7th time, OR is at OR_2, the 2nd time;

and so on.

Thus geometrically we get 5 and only 5 different values for

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}}.$$

EXAMPLES XLVII.

Express the following in the form $r(\cos\theta + i\sin\theta)$:

1. $1+\sqrt{-3}$.
2. $-1+\sqrt{-3}$.
3. $1-\sqrt{-3}$.
4. $4+i.3$.
5. $3+i.17$.

Find the values of

6. $(-1+i\sqrt{3})^{\frac{1}{4}}$.
7. $(65+142\sqrt{-1})^{\frac{1}{5}}$.
8. $\{(2-\sqrt{3})+i\}^{\frac{1}{6}}$.
9. Simplify $\dfrac{(\cos\theta+i\sin\theta)^8}{(\cos\phi+i\sin\phi)^9}$.
10. $\dfrac{(\cos\theta-i\sin\theta)^9}{(\cos\phi+i\sin\phi)^7}$.

11. $\dfrac{(\cos 2\theta + i\sin 2\theta)^{-3}(\cos 3\theta - i\sin 3\theta)^{-4}}{(\cos 4\theta - i\sin 4\theta)^{-5}(\cos 5\theta + i\sin 5\theta)^{-6}}$.

12. $\dfrac{(\cos 3\theta - i\sin 3\theta)^{4}(\cos 4\theta + i\sin 4\theta)^{5}}{(\cos 5\theta + i\sin 5\theta)^{6}(\cos 6\theta - i\sin 6\theta)^{7}}$.

13. Prove

$$\sin m\theta = \sec^m\theta\left\{m\tan\theta - \frac{m(m+1)(m+2)}{\lfloor 3}\tan^3\theta + \ldots\right\}$$

when $$\tan\theta < 1.$$

$$\cos m\theta - i\sin m\theta = (\cos\theta + i\sin\theta)^{-m}.$$

Expand by the Binomial Theorem and equate real and imaginary parts.

14. Show that

$$\left(\cos\frac{\pi}{13} + i\sin\frac{\pi}{13}\right);\quad \left(\cos\frac{3\pi}{13} + i\sin\frac{3\pi}{13}\right);\quad \left(\cos\frac{5\pi}{13} + i\sin\frac{5\pi}{13}\right)\ldots$$
$$\left(\cos\frac{11\pi}{13} + i\sin\frac{11\pi}{13}\right);$$

$$\left(\cos\frac{15\pi}{13} + i\sin\frac{15\pi}{13}\right);\ldots\left(\cos\frac{25\pi}{13} + i\sin\frac{25\pi}{13}\right)$$

are the roots of

$$x^{12} - x^{11} + x^{10} - x^9 + \ldots + 1 = 0.$$

Hence or otherwise show that

$$\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \ldots \cos\frac{11\pi}{13} = \frac{1}{2}.$$

15. From the identity

$$\frac{1}{(x_1 - x_2)(x_1 - x_3)} = \frac{1}{(x_2 - x_3)(x_1 - x_2)} - \frac{1}{(x_2 - x_3)(x_1 - x_3)},$$

prove that

$$\sin(\theta_2 - \theta_3)\cos(2\theta_1 + \theta_2 + \theta_3)$$
$$= \sin(\theta_1 - \theta_3)\cos(\theta_1 + 2\theta_2 + \theta_3) - \sin(\theta_1 - \theta_2)\cos(\theta_1 + \theta_2 + 2\theta_3),$$

where $$x_1 = \cos 2\theta_1 + i\sin 2\theta_1;\quad x_2 = \text{etc.}$$

16. From the identity

$$\frac{(x_1-x_3)(x_1-x_4)}{(x_2-x_3)(x_2-x_4)}+\frac{(x_1-x_4)(x_1-x_2)}{(x_3-x_4)(x_3-x_2)}+\frac{(x_1-x_2)(x_1-x_3)}{(x_4-x_2)(x_4-x_3)}=1,$$

deduce that

$$\frac{\sin(\theta_1-\theta_3)\sin(\theta_1-\theta_4)}{\sin(\theta_2-\theta_3)\sin(\theta_2-\theta_4)}\sin 2(\theta_1-\theta_2)$$
$$+\frac{\sin(\theta_1-\theta_4)\sin(\theta_1-\theta_2)}{\sin(\theta_3-\theta_4)\sin(\theta_3-\theta_2)}\sin 2(\theta_1-\theta_3)$$
$$+\frac{\sin(\theta_1-\theta_2)\sin(\theta_1-\theta_3)}{\sin(\theta_4-\theta_2)\sin(\theta_4-\theta_3)}\sin 2(\theta_1-\theta_4)=0,$$

where $x_1=\cos 2\theta_1+i\sin 2\theta_1;\quad x_2=$ etc.

17. Prove that

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\frac{1}{2},$$

$$\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\frac{1}{2}\sqrt{7}.$$

18. Prove that the continued product of the 4 values of

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} \text{ is } 1.$$

CHAPTER XXIV.

EXPANSIONS FOR SINE AND COSINE OF AN ANGLE IN POWERS OF THE CIRCULAR MEASURE OF THE ANGLE.

230. By De Moivre's Theorem

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + ni\cos^{n-1}\theta\sin\theta$$

$$-\frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta - i\frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta$$

$$+ \ldots\ldots;$$

$\therefore$ equating real and imaginary parts

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta + \ldots\ldots\ldots \text{ (i)},$$

and $\quad \sin n\theta = n\cos^{n-1}\theta\sin\theta - \dfrac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta$

$$+ \ldots\ldots.$$

Put $\quad n\theta = \alpha$ and therefore $n = \dfrac{\alpha}{\theta}$.

Series (i) becomes

$$\cos\alpha = \cos^n\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta} - 1\right)}{\underline{|2}}\cos^{n-2}\theta\sin^2\theta + \ldots\ldots$$

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{\underline{|2}}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \ldots\ldots \text{ (ii)},$$

$$\therefore \frac{(r+1)^{\text{th}} \text{ term}}{r^{\text{th}} \text{ term}}$$

$$= \frac{\dfrac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-\overline{2r-1}\,\theta)}{\lfloor 2r} \cos^{n-2r}\theta \left(\dfrac{\sin\theta}{\theta}\right)^{2r}}{\dfrac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-\overline{2r-3}\,\theta)}{\lfloor 2r-2} \cos^{n-2r+2}\theta \left(\dfrac{\sin\theta}{\theta}\right)^{2r-2}}$$

$$= \frac{(\alpha-\overline{2r-2}\,\theta)(\alpha-\overline{2r-1}\,\theta)}{2r(2r-1)} \left(\frac{\tan\theta}{\theta}\right)^2.$$

If now θ becomes indefinitely small and consequently n indefinitely great, α being constant,

$$\underset{n=\infty}{\text{Lt}} \frac{(r+1)^{\text{th}} \text{ term}}{r^{\text{th}} \text{ term}} = \frac{\alpha^2}{2r(2r-1)}, \quad \text{since } \underset{\theta=0}{\text{Lt}} \left(\frac{\tan\theta}{\theta}\right)^2 = 1,$$

and this limit may be made < 1 by taking r great enough.

Thus series (ii) is convergent since the terms are alternately positive and negative and, after a certain term, each term is greater than the succeeding; moreover

$$\underset{\theta=0}{\text{Lt}}\ r^{\text{th}} \text{ term} = \underset{\theta=0}{\text{Lt}} \frac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-\overline{2r-3}\,\theta)}{\lfloor 2r-2}$$

$$\cos^{n-2r+2}\theta \left(\frac{\sin\theta}{\theta}\right)^{2r-2}$$

$$= \frac{\alpha^{2r-2}}{\lfloor 2r-2} \quad \text{(Art. 220)},$$

it therefore follows that

$$\cos\alpha < 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \ldots\ldots + \frac{\alpha^{4q}}{\lfloor 4q}$$

$$> 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \ldots\ldots - \frac{\alpha^{4q-2}}{\lfloor 4q-2};$$

or

$$\cos\alpha = 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \ldots\ldots - \frac{\alpha^{4q-2}}{\lfloor 4q-2} + \epsilon\frac{\alpha^{4q}}{\lfloor 4q},$$

where ϵ is a proper fraction.

If now q becomes indefinitely great, the series becomes an infinite one and since $\underset{q=\infty}{\text{Lt}} \dfrac{\alpha^{4q}}{\underline{|4q}} = 0$*

$$\cos\alpha = 1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \ldots\ldots \infty.$$

In a similar way it may be proved that

$$\sin\alpha = \alpha - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} - \ldots\ldots \infty.$$

231. It is obvious that each of these series is convergent for the terms are alternately positive and negative, and taking the expansion of $\cos\alpha$ for example, the ratio of the $(r+1)^{\text{th}}$ term to the r^{th} is $\dfrac{\alpha^2}{2r(2r-1)}$ which may be made as small as we please by taking r great enough.

If $\alpha \not> \dfrac{\pi}{4}$ these two series converge very rapidly and five or six terms will give the values of $\sin\alpha$ and $\cos\alpha$ to 7 decimal places.

232. Ex. 1. Calculate to 7 decimal places the value of the sine of an angle whose radian measure is ·5.

$$\sin\alpha = {\cdot}5 - \frac{1}{\underline{|3}}({\cdot}5)^3 + \frac{1}{\underline{|5}}({\cdot}5)^5 - \ldots,$$

$$\begin{aligned} {\cdot}5 &= {\cdot}5, \\ ({\cdot}5)^3 &= {\cdot}125, \\ ({\cdot}5)^5 &= {\cdot}03125, \\ ({\cdot}5)^7 &= {\cdot}0078125, \\ ({\cdot}5)^9 &= {\cdot}001953125 \text{ etc.} \end{aligned}$$

* Suppose $a < c < 4q$ where c is finite and positive.

Then $$\frac{a^{4q}}{\underline{|4q}} = \frac{a^{c-1}}{\underline{|c-1}} \cdot \frac{a \cdot a \cdot a \ldots\ldots a}{c(c+1)(c+2)\ldots\ldots 4q} < \frac{a^{c-1}}{\underline{|c-1}}\left(\frac{a}{c}\right)^{4q-c+1}.$$

Since $a < c$, $$\underset{q=\infty}{\text{Lt}}\left(\frac{a}{c}\right)^{4q-c+1} = 0.$$

Whence required result follows.

$\therefore\ \cdot 5 = \cdot 5,$

$$\frac{1}{\lfloor 5}(\cdot 5)^5 = \cdot 000260416 \qquad \frac{1}{\lfloor 3}(\cdot 5)^3 = \cdot 020833333$$

$$\frac{1}{\lfloor 9}(\cdot 5)^9 = \underline{\cdot 000000005} \qquad \frac{1}{\lfloor 7}(\cdot 5)^7 = \underline{\cdot 000001550}$$

$$\cdot 500260421 \qquad \cdot 020834883.$$

$$\underline{\cdot 020834883}$$

$\therefore\ \sin \cdot 5 = \cdot 4794255$

Ex. 2. Expand $\sin(x+h)$ in powers of h.

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$= \sin x\left(1 - \frac{h^2}{\lfloor 2} + \frac{h^4}{\lfloor 4}\cdots\right) + \cos x\left(h - \frac{h^3}{\lfloor 3} + \frac{h^5}{\lfloor 5} - \cdots\right)$$

$$= \sin x + h\cos x - \frac{h^2}{\lfloor 2}\sin x - \frac{h^3}{\lfloor 3}\cos x + \ldots.$$

Ex. 3. Find (approx.) the number of radians in θ, if

$$\frac{\sin\theta}{\theta} = \frac{5045}{5046}.$$

Since $\dfrac{\sin\theta}{\theta}$ is nearly 1, θ must be small,

$$\therefore\ \frac{\sin\theta}{\theta} = \frac{\theta - \dfrac{\theta^3}{\lfloor 3}}{\theta}\ \text{(approx.)} = 1 - \frac{\theta^2}{\lfloor 3} = \frac{5045}{5046},$$

$$\therefore\ \theta^2 = \tfrac{1}{5046}\cdot\lfloor 3 = \tfrac{1}{841},$$

$$\therefore\ \theta = \tfrac{1}{29}\ \text{radians.}$$

Ex. 4. Find

$$\operatorname*{Lt}_{\theta=0} \frac{\tan 2\theta - 2\tan\theta}{\theta^3}.$$

$$\tan x = \frac{\sin x}{\cos x} = \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}\cdots\right)\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots\right)^{-1}$$

$$= \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}\right)\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4}\right)^{-1},\ \text{if } x \text{ is small}$$

$$= \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5}\right)\left[1 + \left(\frac{x^2}{\lfloor 2} - \frac{x^4}{\lfloor 4}\right) + \left(\frac{x^2}{\lfloor 2}\right)^2\right]\ \text{omitting terms beyond } x^5$$

$$= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5,$$

$$\therefore \frac{\tan 2\theta - 2\tan\theta}{\theta^3} = \frac{\left[2\theta + \frac{1}{3}(2\theta)^3 + \frac{2}{15}(2\theta)^5\right] - 2\left[\theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5\right]}{\theta^3}$$

$$= \frac{2\theta^3 + 4\theta^5}{\theta^3} = 2 + 4\theta^2,$$

$$\therefore \underset{\theta=0}{\text{Lt}} \frac{\tan 2\theta - 2\tan\theta}{\theta^3} = 2.$$

EXAMPLES XLVIII.

1. Find to 7 places of decimals the sine and cosine of 1 radian.

2. Expand $\cos(\alpha + h)$ in powers of h.

3. Find the general term in the expansion of $\cos^3\theta$ in powers of θ.

4. Find the number of radians in θ, if

$$\frac{\sin\theta}{\theta} = \frac{2645}{2646}.$$

5. Find the general term in the expansion of $\sin^3\theta\cos\theta$ in powers of θ.

6. Find the limit of $\{\sin(\alpha+\theta) - \sin\alpha\}/\theta$, when $\theta = 0$.

7. Find the limit of $(\sin^2 3\theta - \sin^2\theta)/(\cos 4\theta - \cos\theta)$, when $\theta = 0$.

8. Find the limiting value of $[\sin(\tan x) - \tan(\sin x)]/x^7$, when $x = 0$.

9. Find the limiting value of $\dfrac{5\sin\theta - \sin 5\theta}{\theta(\cos\theta - \cos 5\theta)}$ when $\theta = 0$.

10. Find the limit when $x = 0$ of

$$\frac{a^2\sin ax - b^2\sin bx}{b^2\tan ax - a^2\tan bx}.$$

11. Find the limiting value when $x = 0$ of

$$\frac{e^x - 1 + \log_e(1+x)}{x}.$$

12. If $\phi = \theta - 2e\sin\theta + \frac{3e^2}{4}\sin 2\theta - \frac{e^3}{3}\sin 3\theta$,

prove that

$$\theta = \phi + 2e\sin\phi + \frac{5e^2}{4}\sin 2\phi + \frac{e^3}{12}(5\sin 3\phi - 3\sin\phi),$$

where powers of e higher than the third are neglected.

13. Prove that when $x = \tan 2\theta$ and θ lies between $-\frac{\pi}{8}$ and $\frac{\pi}{8}$,

$$\tan\theta = \frac{x}{2}\left(1 - \frac{x^2}{4} + \frac{x^4}{8} - \frac{5}{64}x^6 + \dots\right),$$

and that, if powers of x above the 5th are neglected,

$$\sin\theta = \frac{x}{2}\left(1 - \frac{3}{8}x^2 + \frac{31}{128}x^4\right).$$

14. If a, b, c are the sides and $\frac{\pi}{3}+\alpha, \frac{\pi}{3}+\beta, \frac{\pi}{3}+\gamma$ the angles (in circular measure) of a triangle which is very nearly equilateral, so that α, β, γ are very small, prove that approximately

$$a\alpha + b\beta + c\gamma = R(\alpha^2 + \beta^2 + \gamma^2),$$

where R is the radius of the circumscribing circle.

15. Prove that the limit of $\left(\cos\frac{a}{n}\right)^{2n^2}$ is e^{-a^2}, when n is infinite.

16. From the expansion of $\cos\theta$ in terms of θ, prove that

$$\Sigma\frac{(b+c)^{2p}(c+a)^{2q}(a+b)^{2r}}{\underline{|2p}\,\underline{|2q}\,\underline{|2r}} = \frac{a^{2n}+b^{2n}+c^{2n}+(a+b+c)^{2n}}{\underline{|2n}}\,2^{2n-2},$$

where n is a positive integer, and the summation extends to all positive integral values of p, q, r, including zero, such that

$$p + q + r = n.$$

17. If $\cos z = \cos(z+x)\cos\Delta + \sin(z+x)\sin\Delta\cos h$, where x and Δ are so small that higher powers than their cubes may be neglected, prove that

$$x = \Delta\cos h - \tfrac{1}{2}\Delta^2\cot z\sin^2 h + \tfrac{1}{3}\Delta^3\cos h\sin^2 h.$$

18. Express $\sec\theta$ in powers of θ up to θ^6.

19. If θ and ϕ are small angles, prove approximately that

$$\frac{\theta}{\phi} = \frac{2}{3}\frac{\sin\theta}{\sin\phi} + \frac{1}{3}\frac{\tan\theta}{\tan\phi} - \frac{\theta}{180\phi}(\theta^2-\phi^2)(9\theta^2-\phi^2).$$

20. Assuming the expansion of $\sin\theta$ in powers of θ, prove that

$$\theta = \sin\theta + \frac{1}{2}\frac{\sin^3\theta}{3} + \frac{1.3}{2.4}\frac{\sin^5\theta}{5} + \dots.$$

21. If $\sin(30° + \theta) = {\cdot}51$, prove that $\theta = 39' 50''$ (approx.).

TEST PAPERS.

[*Including Properties of Triangles. Chapters XIII and XI*

XLVI.

1. Prove that the radii of the circles inscribed in t triangles into which ABC is divided by the line which bise the angle A, are to one another in the ratio

$$\cos\frac{C}{2}\left\{1+\tan\frac{C-B}{4}\right\} : \cos\frac{B}{2}\left\{1+\tan\frac{B-C}{4}\right\}.$$

2. Show that in a triangle

$$a^2\cos 2B + b^2\cos 2A = a^2 + b^2 - 4ab\sin A\sin B.$$

3. Prove

$$\tan^2 A - \tan^2 B = \frac{\cos^2 B - \cos^2 A}{\cos^2 A\cos^2 B}.$$

4. In a triangle ABC, I is the centre of the inscribed circ ID is perpendicular to BC, BM and CN are perpendicular to A show that the triangles MDB and DNC are equiangular and hen prove geometrically that $bc\sin^2\frac{A}{2} = (s-b)(s-c)$.

5. Show how to construct the triangle ABC when r, R a the angle A are given, and establish the limitation that the ra of r to R must not be greater than

$$2\sin\frac{A}{2}\left(1-\sin\frac{A}{2}\right).$$

6. Given that $\tan A$ and $\tan B$ are the roots of
$$x^2+px+q=0,$$
find the value of
$$\sin^2(A+B)+p\,.\sin(A+B)\cos(A+B)+q\cos^2(A+B).$$

7. Given
$$\theta+\phi=\alpha \text{ and } \sin^2\theta-\sin^2\phi=k,$$
prove that
$$\sin(\theta-\phi)=\frac{k}{\sin\alpha}.$$

XLVII.

1. O is a point within a triangle ABC such that
$$CAO=ABO=BCO=\alpha,$$
prove that
$$\cot\alpha=\cot A+\cot B+\cot C.$$

2. Prove that
$$\frac{1}{r}=\frac{\cos\frac{A}{2}}{a\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\cos\frac{B}{2}}{b\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\cos\frac{C}{2}}{c\cos\frac{A}{2}\cos\frac{B}{2}}.$$

3. P, Q, R are three successive milestones on a straight road. A is a point such that $A\hat{P}Q=20°$; $A\hat{R}Q=30°$. Find AP in yards.

4. In a circle 5 metres radius what is the length of the arc which subtends an angle 33° 15′ at the centre? $(\pi=\frac{22}{7}.)$

5. ABCD is a quadrilateral, AB = 147, AC = 136, AD = 98 metres.
$$B\hat{A}C=22°\,30'\,;\quad C\hat{A}D=34°\,15'.$$ Find the area.

6. Prove that

(i) $$\tan^4\theta=\frac{2\tan\theta-\sin 2\theta}{2\cot\theta-\sin 2\theta}.$$

(ii) $$\tan\theta+\cot\frac{\theta}{2}-\cot\frac{\theta}{2}\sec\theta=0.$$

7. In a triangle

$$2\cos A + \cos B + \cos C = 2,$$

prove that

$$2a = b + c.$$

XLVIII.

1. If

$$\tan(B - C) = \frac{3\sin 2C}{5 - 3\cos 2C},$$

prove that

$$\tan B = 4\tan C.$$

2. Prove that the medians of a triangle form with the side they bisect 3 angles such that the sum of their cotangents is zero when the angles are measured in the same sense of rotation.

3. If Q and R are the points of trisection of the side BC of a triangle ABC, prove that

$$\sin B\hat{A}R \,.\, \sin C\hat{A}Q = 4\sin B\hat{A}Q \,.\, \sin C\hat{A}R,$$

and

$$(\cot BAQ + \cot QAR)(\cot CAR + \cot RAQ) = 4\operatorname{cosec}^2 QAR.$$

4. Prove

(i) $$\frac{1 + \tan^2\left(\frac{\pi}{4} - \theta\right)}{1 - \tan^2\left(\frac{\pi}{4} - \theta\right)} = \operatorname{cosec} 2\theta.$$

(ii) $$\frac{1}{2}\left(\cot\frac{\theta}{2} - \tan\frac{\theta}{2}\right) = \cot\theta.$$

5. Solve a triangle, given

$$a = 2143;\ c = 4172;\ A = 25^\circ\, 1'.$$

6. The nautical mile is an arc of the earth's equator which subtends an angle 1′ at the centre; find its length correct to the nearest foot, using

one radian = 206265″; earth's equatorial radius = 20926000 ft.

7. Prove that

$$\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3}.$$

L.

1. With the ordinary notation prove

$$\Delta = 2R^2 \sin A \sin B \sin C = r^2 \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

2. If the angle A of a triangle is 60°, prove that

$$(a+b+c)(-a+b+c) - 3bc = 0.$$

3. In a triangle in which $a+b=2c$, prove that

$$a\cos B - b\cos A = 2a - 2b.$$

4. In the side CA of a triangle ABC a point A′ is taken and in CB produced B′ is taken so that A′B and AB′ are parallel, prove

$$\frac{AB^2}{AB' \cdot A'B} = \frac{\sin A' \sin B'}{\sin A \sin B}.$$

5. Prove that

(i) $$\frac{\sin 7\theta + \sin 5\theta}{\cos 5\theta - \cos 7\theta} = \cot\theta.$$

(ii) $$\cos 7\theta - \cos 13\theta = 2\{\sin 11\theta \sin 2\theta + \sin 7\theta \sin 2\theta + \cos 6\theta \cos\theta - \cos 5\theta\}.$$

6. Given

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{1+\tan\alpha\tan\beta},$$

prove that

$$\sin 2\theta = \frac{\sin 2\alpha + \sin 2\beta}{1+\sin 2\alpha \sin 2\beta}.$$

7. In a triangle ABC, b and c are given and it is known that the height AD = the base BC, prove that

$$\frac{c}{2}(\sqrt{5}-1) < b < \frac{c}{2}(\sqrt{5}+1).$$

LI.

1. Prove that

$$\frac{(\sec\theta-\tan\theta)^2-1}{(\sec\theta+\tan\theta)^2-1}=\frac{\sin\theta-1}{\sin\theta+1}.$$

2. If I be the centre of the circle inscribed in a triangle ABC show that

$$\frac{r}{s}=\frac{\text{AI . BI . CI}}{abc}.$$

3. Prove that

$$\cos 2\alpha+\cos 4\alpha+\cos 6\alpha+\cos 8\alpha=4\cos 5\alpha\cos 2\alpha\cos\alpha.$$

4. In any triangle show that

$$\frac{b^2-c^2}{\tan \text{A}}+\frac{c^2-a^2}{\tan \text{B}}+\frac{a^2-b^2}{\tan \text{C}}=0.$$

5. Given that $\qquad \cos\theta=\dfrac{\cos\phi-e}{1-e\cos\phi},$

prove that $\qquad \tan\dfrac{\theta}{2}=\tan\dfrac{\phi}{2}\sqrt{\dfrac{1+e}{1-e}}.$

6. If $\tan\theta=\dfrac{b}{a}$, prove that

$$a\cos 2\theta+b\sin 2\theta=a.$$

7. In the triangles ABC, A′B′C′ the angles B and B′ are equal, while the angles A and A′ are supplementary; show that

$$aa'=bb'+cc'.$$

[*Including General Values of Equations and Inverse Functions. Chapters XVI and XVIII.*]

LII.

1. Find the length of an arc on the sea which subtends an angle of one minute at the centre of the earth, supposing the earth a sphere of diameter 7920 miles.

2. Solve $\sin 2\theta = \cos 3\theta$.

3. If $A + B + C =$ an odd multiple of π, show that

$$\sin^2 B + \sin^2 C = \sin^2 A + 2 \cos A \sin B \sin C.$$

4. ABC is a triangle in a horizontal plane, with a right angle at C, and P is the middle point of AB; a flagstaff is set up at C and it is found that its angles of vertical elevation at A, B and P are α, β, γ; show that

$$\tan^2 \gamma = 2 \tan \alpha \tan \beta \sin 2A.$$

5. The difference between the perimeters of an inscribed and a circumscribed regular dodecagon equals a; show that the difference between their areas equals

$$\frac{a^2}{192\left(1 - \cos\frac{\pi}{12}\right)^2}.$$

6. Solve the equation

$$\tan \theta = \frac{1}{6} \cdot \frac{\sin 2\theta}{\cos 2\theta - \frac{1}{3}}.$$

7. Prove that

$$2 \tan^{-1} \tfrac{1}{8} + \tan^{-1} \tfrac{1}{7} + 2 \tan^{-1} \tfrac{1}{5} = 45°.$$

LIII.

1. Solve the equation

$$2 \sin^2 x = \cos^2 \frac{3x}{2}.$$

2. Prove that

(i) $\tan^{-1} 3 + \tan^{-1} 2 + \tan^{-1} 1 = \pi$.

(ii) $\sin^{-1} \frac{5}{13} + \tan^{-1} \frac{7}{24} = \cos^{-1} \frac{253}{325}$.

3. AB is a horizontal road 1 kilometre long running S.E. from A to B. At A a balloon is observed due E. at an elevation of 58°15′, and at B it is seen in a direction N. 27°12′ E. Find the height of the balloon to the nearest metre.

4. In any triangle, prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}.$$

5. From the top of a vertical tower which stands on a flat plain, a length a of a flagstaff projects, and is inclined at an angle γ to the horizon. At a point on the ground, in the vertical plane containing the tower and flagstaff, the elevations of the top of the tower and of the end of the flagstaff are found to be α, β respectively; prove that the height of the tower is

$$a\sin\alpha\sin(\beta+\gamma)\operatorname{cosec}(\beta-\alpha).$$

6. If P is a point in the side BC of a triangle such that $m\text{BP}=n\text{CP}$, show that

$$mc^2+nb^2=(m+n)\text{AP}^2+m\text{BP}^2+n\text{CP}^2,$$

and deduce that

$$\text{AP}^2=\frac{(m+n)(mc^2+nb^2)-mna^2}{(m+n)^2}.$$

7. Prove that

$$\tan\theta+2\tan 2\theta=\cot\theta-4\cot 4\theta.$$

LIV.

1. If $A+B+C=180°$, prove that

$$\sin A\sec\tfrac{1}{2}A+(\sin B+\sin C)\tan\tfrac{1}{2}A=(\sin B-\sin C)\cot\tfrac{1}{2}(B-C).$$

2. Solve the equations

(i) $\cot A-\operatorname{cosec} 2A=1$,

(ii) $\cos^3 A\sin 3A+\sin^3 A\cos 3A=\dfrac{3\sqrt{3}}{8}$.

3. Prove that

$$\tan^{-1}x=2\tan^{-1}\left[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x\right].$$

4. From two points A and B 150 metres apart in a horizontal plane, the line joining the foot of a tower, in the same plane, to B subtends an angle of 97° 13′ at A, and that joining the foot of the tower to A subtends 22° 20′ at B. Find the height of the tower, if the angle of elevation at A is 37° 10′. Answer to the nearest decimetre.

5. Find a value of x which satisfies the equation

$$4\cos x + 5\sin x = 5\cdot 2.$$

6. Find the area of a triangle in which the two sides are 187·5 and 925·8 centimetres, and the included angle 27°15′.

7. If I is the centre of the inscribed circle of a triangle ABC, show that the radius of the circle inscribed in the triangle BIC is

$$\sqrt{2}a \frac{\sin\frac{B}{4}\sin\frac{C}{4}}{\cos\frac{A}{4} - \sin\frac{A}{4}}.$$

LV.

1. Solve

$$\tan^{-1}(x-1) + \tan^{-1}(2-x) = 2\tan^{-1}\sqrt{3x - x^2 - 2}.$$

2. In any triangle show that

$$abc(1 - 2\cos A\cos B\cos C)$$
$$= a^3\cos B\cos C + b^3\cos C\cos A + c^3\cos A\cos B.$$

3. Solve

$$\sin\theta - 2\sin 2\theta\cos\theta + \cos 3\theta = \sin 3\theta.$$

4. Show that

$$\cos^{-1}\frac{4}{5} - \sin^{-1}\frac{1}{\sqrt{10}} + \tan^{-1}\frac{1}{2} = 45°.$$

5. Draw a line BC and divide it at N so that NC = 2BN; draw AN at right angles to BC and equal to BN; join AB, AC and show that

$$2(\tan A + \tan B + \tan C) + 3 = 0.$$

6. OD, OE and OF are the perpendiculars from a point O to the sides of a triangle, show that

$$\cot ADC + \cot BEA + \cot CFB = 0.$$

7. Show that

(i) $\cos\tan^{-1}x = \dfrac{1}{\sqrt{1+x^2}}.$

(ii) $\tan\cos^{-1}x = \dfrac{\sqrt{1-x^2}}{x}.$

LVI.

1. If k is the length of the bisector of the angle A of a triangle ABC, prove that

$$\Delta = \frac{1}{2} k (b + c) \sin \frac{A}{2} = \frac{1}{2} ka \cos \frac{B - C}{2}.$$

2. Find all the values of $\tan \theta$ consistent with

(i) $\cos 4\theta = \cdot 8049$.

(ii) $\tan (\pi \cot \theta) = \cot (\pi \tan \theta)$.

3. Prove that

$$\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5} = \frac{\pi}{4}.$$

4. Solve

$$\tan^{-1} (ax + b) + \tan^{-1} (ax - b) = \frac{\pi}{4}.$$

5. ABC is an equilateral triangle and P is a point in BC such that $PB = \frac{1}{4} BC$; show that $B\hat{A}P = 13° 54'$.

6. If $A + B + C = 180°$, show that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$$

7. With the usual notation of inscribed and escribed circles, show that

$$rr_1 (r_3 - r_2) = (b - c)\, r_2 r_3 \tan \frac{A}{2}.$$

LVII.

1. Show that in any triangle

(i) $\cos A + \cos B = \dfrac{a + b}{c} \cdot 2 \sin^2 \dfrac{C}{2}$.

(ii) $a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B) = 3abc$.

2. Prove that

$$\frac{\operatorname{cosec}\theta\operatorname{cosec}\frac{\phi}{2}-\operatorname{cosec}\phi\operatorname{cosec}\frac{\theta}{2}}{\operatorname{cosec}\theta\operatorname{cosec}\frac{\phi}{2}+\operatorname{cosec}\phi\operatorname{cosec}\frac{\theta}{2}}=\tan\frac{\theta+\phi}{4}\tan\frac{\theta-\phi}{4}.$$

3. ABC is a triangle and P is a point within the angle A, such that A and P are on opposite sides of BC. If CP subtends an angle α at A and β at B, show that

$$\text{PB}\sin(\text{C}-\beta+\alpha)=c\sin(\text{A}-\alpha).$$

4. In a triangle ABC the circum-radius is n times the in-radius, prove that

$$\frac{2abc(n+1)}{n}=a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c).$$

5. If $\tan(2\alpha-3\beta)=\cot(3\alpha-2\beta)$,

and $\tan(2\alpha+3\beta)=\cot(3\alpha+2\beta)$,

show that α and β are both multiples of $\frac{\pi}{10}$.

6. Show that

$$2\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=\frac{\pi}{4}-\tan^{-1}\frac{6}{61}.$$

7. Show that

$$\tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)-\tan^{-1}\left(\frac{x-\cos\alpha}{\sin\alpha}\right)=\frac{\pi}{2}-\alpha.$$

MISCELLANEOUS EXAMPLES.

1. Prove that

(i) $\cos^2 A + \cos^2 (120^\circ + A) + \cos^2 (120^\circ - A) = \frac{3}{2}$.

(ii) $\cos^2 (A - 45^\circ) + \cos^2 A + \cos^2 (A + 45^\circ) + \cos^2 (A + 90^\circ) = 2$.

2. In the ambiguous case of the solution of a triangle when a, b, A are given, prove that

$$\text{(i)} \quad c_1 + c_2 = 2b \cos A,$$

$$\text{(ii)} \quad c_1 \sim c_2 = 2a \cos B.$$

3. The lengths of two adjacent sides of a parallelogram are a and b, and their included angle is α; show that the area of the parallelogram formed by the bisectors of the interior angles is $\frac{1}{2}(a - b)^2 \sin \alpha$.

4. The elevation of the top of a flagstaff on the summit of a hill is observed to be α. When the observer walks a distance a directly towards the hill, the top of the flagstaff is found to have an elevation β, while the elevation of the hill top is γ. Show that the height of the hill is

$$a \sin \alpha \cos \beta \tan \gamma \operatorname{cosec} (\beta - \alpha).$$

5. Prove that

$$\sin 2(B - C) + \sin 2(C - A) + \sin 2(A - B)$$
$$= -4 \sin (B - C) \sin (C - A) \sin (A - B).$$

6. Show that $\quad 5 \sin x + 3 \sin (x + 60^\circ) \not> 7$.

7. Solve $\quad \cos 6\theta + \cos 4\theta = \sin 3\theta + \sin \theta$.

8. Prove that

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$

9. If $y = a\sin x + b\cos x$, express y in the form $A\sin(x+\alpha)$, where A, α are independent of x; and hence show that y must lie in value between $\pm(a^2+b^2)^{\frac{1}{2}}$.

10. Prove that

$$abc(a\cos A + b\cos B + c\cos C) = 8\Delta^2.$$

11. A man travelling along a straight road on a plane observes the angle of elevation of the top of a hill as he passes three successive kilometre-stones to be α, α, β respectively. Prove that the height of the hill is

$$1000\,\{2\operatorname{cosec}(\alpha+\beta)\operatorname{cosec}(\alpha-\beta)\}^{\frac{1}{2}}\sin\alpha\sin\beta \text{ metres.}$$

12. Prove that

$$\sec A = \frac{\cos\frac{1}{2}A}{\sqrt{1+\sin A}} + \frac{\sin\frac{1}{2}A}{\sqrt{1-\sin A}}.$$

13. In any triangle prove that

$$\frac{\cos A}{c\sin B} + \frac{\cos B}{a\sin C} + \frac{\cos C}{b\sin A} = \frac{1}{R}.$$

14. If $\tan(\theta+\alpha) - \tan(\theta-\alpha) = \dfrac{2\tan\theta}{\cos^2\theta - \sin^2\theta\tan^2\alpha}$,

prove that $\theta = \frac{1}{2}n\pi + \alpha$ or $(m+\frac{1}{2})\pi - \alpha$.

15. If α, β are two angles, not differing by 0 or a multiple of 2π, which satisfy the equation $a\cos x + b\sin x = 1$, then will

$$a = \cos\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta), \quad b = \sin\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta).$$

16. If P is a point in the side BC of a triangle, such that $m\,.\,BP = n\,.\,CP$, show that

$$\frac{\sin B\hat{A}P}{\sin C\hat{A}P} = \frac{nb}{mc}.$$

17. Prove that

$$\cot\left(\frac{\pi}{4}+\theta\right) = \sec 2\theta - \tan 2\theta.$$

18. Show that $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ have the same sign only when $\sin\alpha$ numerically exceeds $\sin\beta$.

19. In any triangle, prove that

$$\tan(A-45^\circ)+\tan(B-60^\circ)+\tan(C-75^\circ) = \tan(A-45^\circ)\tan(B-60^\circ)\tan(C-75^\circ).$$

20. Prove that

$$8\sin 10^\circ \sin 40^\circ \sin 80^\circ = 2\cos 20^\circ - 1.$$

21. In any triangle, prove that

$$a\cos(B-C)+b\cos(C+A)+c\cos(A+B)=0.$$

22. If $\cos\theta+\cos\phi=a$ and $\sin\theta+\sin\phi=b$, prove that

$$\sin(\theta+\phi)=\frac{2ab}{a^2+b^2}.$$

23. From a window on one side of a street, a building on the other side is observed to subtend an angle α. If the width of the street be a feet, and if the height of the point of observation be h feet, show that the height of the building is

$$\frac{(a^2+h^2)\sin\alpha}{a\cos\alpha+h\sin\alpha}.$$

24. Solve $\quad \sin 2\theta+\cos 2\theta=\sin\theta+\cos\theta.$

25. In any triangle, if

$$\cos\theta=\frac{a}{b+c},\quad \cos\phi=\frac{b}{c+a},\quad \cos\psi=\frac{c}{a+b},$$

prove $\quad \tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi\tan\tfrac{1}{2}\psi=\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C.$

26. Prove that

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{3\pi}{9}\tan\frac{4\pi}{9}=3.$$

27. Prove that

$$\tan(45^\circ-\theta)\sin 4\theta=[\cos 2\theta+\sin 2\theta-1][\cos 2\theta-\sin 2\theta+1].$$

28. Two chords of a circle, subtending angles 2α, 2β at the centre O, intersect in a point E within the circle; prove that if θ be the angle between them, and r the radius of the circle, the distance $OE = r\operatorname{cosec}\theta(\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\theta)^{\frac{1}{2}}$, it being supposed that the centre is not within the θ angular space.

XLIX.

1. Solve a triangle, given

$$a = 9621;\ b = 6753;\ A = 69°41'.$$

2. Prove that $a \sin\theta + b\cos\theta$ lies between

$$\sqrt{a^2+b^2};\ -\sqrt{a^2+b^2},$$

for all values of θ; also that

$$a \sin^2\theta + 2h\sin\theta\cos\theta + b\cos^2\theta$$

lies between

$$\frac{a+b}{2} + \sqrt{h^2 + \tfrac{1}{4}(a-b)^2},$$

and
$$\frac{a+b}{2} - \sqrt{h^2 + \tfrac{1}{4}(a-b)^2}.$$

3. Prove that

$$\frac{AI}{AI_1} + \frac{BI}{BI_2} + \frac{CI}{CI_3} = 1,$$

where ABC is a triangle and I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles.

4. In any triangle, prove that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r} = \frac{s}{r_1} + \frac{s}{r_2} + \frac{s}{r_3},$$

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \frac{(s-a)(s-b)(s-c)}{r^3} = \frac{s^3}{r_1r_2r_3}.$$

5. A quadrilateral of perimeter $2s$ inscribed in a circle has two opposite vertices at the ends of a diameter. If a, b are two sides on the same side of the diameter, show that the area of the quadrilateral is $(s-a)(s-b)$.

6. Prove that

(i) $\cos A - \cos 2A = 6\sin^2\frac{A}{2} - 8\sin^4\frac{A}{2}$.

(ii) $\operatorname{cosec} A + 2\operatorname{cosec} 2A = \sec A \cot\frac{A}{2}$.

7. The sides of a triangle are p, q, and $\sqrt{p^2+pq+q^2}$, find the greatest angle.

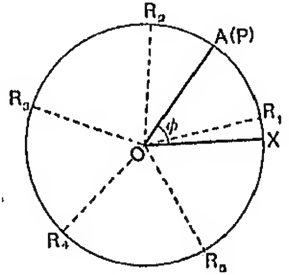
We will do so for the case

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}}$$

the method being perfectly general.

Let a line OP starting from OX, revolve positively. Every time it passes OA it represents

$$(\cos\phi + i\sin\phi).$$



Let OR revolve $\frac{1}{5}$ as fast, then from the above the position of OR at the instant OP passes OA will indicate one value of

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}}.$$

(i) When OP is at OA the 1st time, OR is at OR_1,

$X\hat{O}R_1 = \frac{\phi}{5}$, and $(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \cos\frac{\phi}{5} + i\sin\frac{\phi}{5}$;

(ii) when OP is at OA the 2nd time, OR is at OR_2,

$XOR_2 = \frac{2\pi + \phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{2\pi + \phi}{5} + i\sin\frac{2\pi + \phi}{5}\right)^{\frac{1}{5}};$$

(iii) when OP is at OA the 3rd time, OR is at OR_3,

$XOR_3 = \frac{4\pi + \phi}{5}$, and

$$(\cos\phi + i\sin\phi)^{\frac{1}{5}} = \left(\cos\frac{4\pi + \phi}{5} + i\sin\frac{4\pi + \phi}{5}\right)^{\frac{1}{5}};$$

Again $$r_1 r_2 = r_2 r_1$$

and $$\theta_1 + \theta_2 = \theta_2 + \theta_1;$$

$$\therefore (r_1 r_2, \ \theta_1 + \theta_2) = (r_2 r_1, \ \theta_2 + \theta_1),$$

$$\therefore (r_1, \theta_1) \times (r_2, \theta_2) = (r_2, \theta_2) \times (r_1, \theta_1).$$

Thus complex quantities when multiplied obey the Commutative Law.

228. By Art. 227

$$\{r, \ \theta\}^2 = (r^2, 2\theta),$$

$$\therefore \text{ one of the values of } \{r^2, 2\theta\}^{\frac{1}{2}} \text{ is } (r, \theta).$$

$$\therefore \text{ one of the values of } (1, \pi)^{\frac{1}{2}} \text{ is } \left(1, \frac{\pi}{2}\right).$$

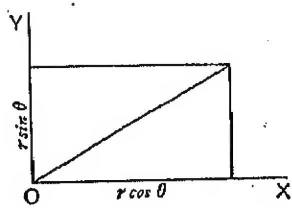
But $(1, \pi)$ is what is usually called -1.

$$\therefore \left(1, \frac{\pi}{2}\right) \text{ is what is usually called } \sqrt{-1} \text{ or } i.$$

Thus if OY is perpendicular to OX, unit length along OY represents i.

$\therefore$ a length $r \sin\theta$ along OY represents $i \, . \, r\sin\theta$.

Thus



$$\left(r\sin\theta, \frac{\pi}{2}\right) = i \, . \, r\sin\theta,$$

$$(r\cos\theta, \ 0) = r\cos\theta.$$

But by Art. 226

$$(r\cos\theta, \ 0) + \left(r\sin\theta, \frac{\pi}{2}\right) = (r, \theta),$$

$$\therefore \ r(\cos\theta + i\sin\theta) = (r, \theta).$$

Thus the figure in Art. 227 is the geometrical representation of the identity

$$r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}.$$

Now

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$$
$$\equiv (a+b+c)(a-b-c)(a-b+c)(a+b-c)=0,$$

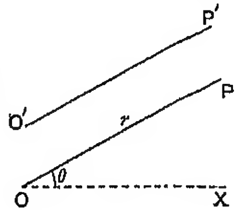
$$\therefore\ \Sigma(\cos\alpha+i\sin\alpha)^4-2\Sigma(\cos\beta+i\sin\beta)^2(\cos\gamma+i\sin\gamma)^2=0,$$
$$\Sigma(\cos 4\alpha+i\sin 4\alpha)-2\Sigma(\cos 2\beta+i\sin 2\beta)(\cos 2\gamma+i\sin 2\gamma)=0,$$
$$\Sigma(\cos 4\alpha+i\sin 4\alpha)-2\Sigma\{\cos 2(\beta+\gamma)+i\sin 2(\beta+\gamma)\}=0.$$

Equating real parts,

$$\Sigma\cos 4\alpha=2\Sigma\cos 2(\beta+\gamma).$$

225. Geometrical representation of complex quantities.

The position of a point P relative to O is defined by the direction and length of the line $\overline{\text{OP}}$.



$\overline{\text{OP}}$ here indicates not merely a line but the operation of moving a point from O to P in the direction of the line OP.

$\overline{\text{OP}}$ is called a Geometrical Vector.

A complex quantity may be represented by a geometrical vector.

The length of OP (r) is called the *Modulus*.

The angle (θ) between OP and a standard direction OX is called the *Amplitude*.

OX is called the *Primary* axis.

The complex quantity is written (r, θ).

Since we have no conception of absolute position the vector or complex quantity

$$\overline{\text{O}'\text{P}'}\equiv\overline{\text{OP}},$$

when O′P′ is geometrically parallel to OP and equal to it in length, *i.e.* when OP and O′P′ are the opposite sides of a parallelogram.

$$\therefore \frac{h\theta}{k} + \frac{2r\pi}{k} - \left(\frac{h\theta}{k} + \frac{2s\pi}{k}\right) = \text{a multiple of } 2\pi,$$

i.e. $$\frac{2\pi}{k}(r-s) = \text{a multiple of } 2\pi,$$

i.e. $$r - s = \quad \text{,,} \quad \text{of } k,$$

which is impossible when both r and s are limited to

$$0, 1, 2, 3, \ldots\ldots, (k-1).$$

Thus we have found k different values of

$$(\cos h\theta + i \sin h\theta)^{\frac{1}{k}},$$

i.e. of $$(\cos\theta + i\sin\theta)^{\frac{h}{k}},$$

and by the Theory of Equations $x^k = c$ has only k roots, *i.e.* no k^{th} root of a quantity can have more than k values.

224. Ex. 1. To extract the nth root of $a + ib$.

1st, put $a + ib$ in the form $r(\cos\theta + i\sin\theta)$.

Thus let $$r\cos\theta = a, \quad r\sin\theta = b;$$

so that $$r^2 = a^2 + b^2, \quad \tan\theta = \frac{b}{a};$$

$$(a+ib)^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos\theta + i\sin\theta)^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right),$$

or $$r^{\frac{1}{n}}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right),$$

or $$r^{\frac{1}{n}}\left(\cos\frac{\theta+4\pi}{n} + i\sin\frac{\theta+4\pi}{n}\right),$$

....

....

or $$r^{\frac{1}{n}}\left\{\cos\frac{\theta+\overline{2n-2}\pi}{n} + i\sin\frac{\theta+\overline{2n-2}\pi}{n}\right\},$$

and by substituting for r and θ we thus have the n, nth roots of $a + ib$.

$$\therefore \operatorname*{Lt}_{n=\infty}\left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n \text{ lies between } 1^n \text{ (or 1) and } \operatorname*{Lt}_{n=\infty}\left(\cos\frac{\alpha}{n}\right)^n;$$

$$\therefore \operatorname*{Lt}_{n=\infty}\left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n = 1.$$

EXAMPLES XLVI.

Prove that:

1. $\dfrac{2}{\lfloor 3} + \dfrac{4}{\lfloor 5} + \dfrac{6}{\lfloor 7} + \ldots = e^{-1}.$

2. $\dfrac{2}{\lfloor 1} + \dfrac{4}{\lfloor 3} + \dfrac{6}{\lfloor 5} + \ldots = e.$

3. $e + \dfrac{1}{e} = 2\left[1 + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 4} + \dfrac{1}{\lfloor 6} + \ldots\right].$

4. $e - \dfrac{1}{e} = 2\left[1 + \dfrac{1}{\lfloor 3} + \dfrac{1}{\lfloor 5} + \dfrac{1}{\lfloor 7} + \ldots\right].$

5. $\dfrac{e+1}{e-1} = \dfrac{1 + \dfrac{1}{\lfloor 3} + \dfrac{1}{\lfloor 5} + \ldots}{\dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 4} + \dfrac{1}{\lfloor 6} + \ldots}.$

6. $5e = 1 + \dfrac{2^3}{\lfloor 2} + \dfrac{3^3}{\lfloor 3} + \dfrac{4^3}{\lfloor 4} + \ldots.$

7. $\operatorname*{Lt}_{n=\infty}\left(1 + \dfrac{x}{n}\right)^{\frac{n}{y}} = e^{\frac{x}{y}}.$

8. From identity $x = \dfrac{1 - \dfrac{1}{x+1}}{1 - \dfrac{x}{x+1}},$

prove $\log_e x = \dfrac{x-1}{x+1} + \dfrac{x^2-1}{2(x+1)^2} + \dfrac{x^3-1}{3(x+1)^3} + \ldots.$

Ex. 6. Find the value of

$$\tan\alpha + \frac{1}{2}\tan\frac{\alpha}{2} + \frac{1}{2^2}\tan\frac{\alpha}{2^2} + \ldots \text{ to } n \text{ terms.}$$

$$\tan\alpha = \cot\alpha - 2\cot 2\alpha$$

$$\frac{1}{2}\tan\frac{\alpha}{2} = \frac{1}{2}\cot\frac{\alpha}{2} - \cot\alpha$$

$$\frac{1}{2^2}\tan\frac{\alpha}{2^2} = \frac{1}{2^2}\cot\frac{\alpha}{2^2} - \frac{1}{2}\cot\frac{\alpha}{2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2^{n-1}}\tan\frac{\alpha}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{\alpha}{2^{n-2}};$$

therefore, adding $\qquad S = \dfrac{1}{2^{n-1}}\cot\dfrac{\alpha}{2^{n-1}} - 2\cot 2\alpha.$

Series involving the squares and cubes of sines and cosines may be evaluated by transforming them into new series containing multiple angles.

Ex. 7. Find the value of

$$\cos^2\alpha + \cos^2(\alpha+\beta) + \cos^2(\alpha+2\beta) + \ldots \text{ to } n \text{ terms.}$$

Since $\qquad 2\cos^2\alpha = 1 + \cos 2\alpha,$

$$\therefore\ 2S = \{1+\cos 2\alpha\} + \{1+\cos 2(\alpha+\beta)\} + \{1+\cos 2(\alpha+2\beta)\} + \ldots$$

$$= n + \cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta) + \ldots$$

$$= n + \frac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{\sin\beta}.$$

Ex. 8. Find the value of

$$\sin^3\alpha + \sin^3 3\alpha + \sin^3 5\alpha + \ldots \text{ to } n \text{ terms.}$$

Since $\qquad 4\sin^3\alpha = 3\sin\alpha - \sin 3\alpha,$

$$\therefore\ 4S = (3\sin\alpha - \sin 3\alpha) + (3\sin 3\alpha - \sin 9\alpha) + (3\sin 5\alpha - \sin 15\alpha) + \ldots$$

$$= 3(\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots) - (\sin 3\alpha + \sin 9\alpha + \sin 15\alpha + \ldots)$$

$$= \frac{3\sin^2 n\alpha}{\sin\alpha} - \frac{\sin^2 3n\alpha}{\sin 3\alpha}.$$

210. If in the sine-series, we change β into $\beta+\pi$, we obtain

$$\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{\sin\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\cos\frac{\beta}{2}},$$

and in the cosine-series

$$\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{\cos\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\cos\frac{\beta}{2}}.$$

211. Ex. 1. Find the value of $\sin A+\sin 3A+\sin 5A+\ldots$ to n terms.

By Art. 206,

$$\text{Series}=\frac{\sin\left(A+\frac{n-1}{2}\,2A\right)\sin n\frac{2A}{2}}{\sin\frac{2A}{2}}$$

$$=\frac{\sin^2 nA}{\sin A}.$$

Ex. 2. Find the value of $\cos\frac{\pi}{17}+\cos\frac{3\pi}{17}+\ldots+\cos\frac{15\pi}{17}$.

By Art. 207,

$$\text{Series}=\frac{\cos\left(\frac{\pi}{17}+\frac{7}{2}\cdot\frac{2\pi}{17}\right)\sin\frac{8\pi}{17}}{\sin\frac{\pi}{17}}$$

$$=\frac{\cos\frac{8\pi}{17}\sin\frac{8\pi}{17}}{\sin\frac{\pi}{17}}=\frac{1}{2}\,\frac{\sin\frac{16\pi}{17}}{\sin\frac{\pi}{17}}=\frac{1}{2}.$$

22. $\sin^{-1}\left(\frac{x-a+b}{2b}\right)^{\frac{1}{2}}=\frac{1}{2}\cos^{-1}\left(\frac{a-x}{b}\right)$.

23. $\cot^{-1}\frac{ab+1}{a-b}+\cot^{-1}\frac{bc+1}{b-c}+\cot^{-1}\frac{ca+1}{c-a}=0$.

24. $2\cos^{-1}\sqrt{\frac{a-x}{a-b}}=\sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{a-b}$.

25. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

26. $\tan^{-1}x+\tan^{-1}\frac{1-x}{1+x}=\frac{\pi}{4}$.

27. $\sin^{-1}\frac{\sqrt{5}}{5}+\cot^{-1}3=\frac{\pi}{4}$.

28. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-yz-zx-xy}\right)$.

29. $x^2=\sin 2y$, when

$$y=\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}.$$

Solution of equations.

100. This is best illustrated by actual examples.

(i) Solve

$$\tan^{-1}x+\tan^{-1}(1-x)=2\tan^{-1}\sqrt{x-x^2}.$$

From Art. 194

$$\tan^{-1}\frac{x+(1-x)}{1-x(1-x)}=\tan^{-1}\frac{2\sqrt{x-x^2}}{1-(x-x^2)};$$

$$\therefore \quad \frac{1}{1-x+x^2}=\frac{2\sqrt{x-x^2}}{1-x+x^2};$$

therefore either

$$1-x+x^2=0, \text{ i.e. } x=\frac{1\pm\sqrt{-3}}{2},$$

or

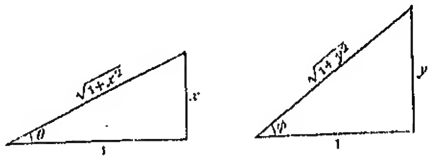
$$2\sqrt{x-x^2}=1, \text{ i.e. } x=\frac{1}{2}.$$

Ans. $\frac{1}{2}$ or $\frac{1\pm\sqrt{-3}}{2}$.

194. (ii) *Find the value of*

$$\tan^{-1} x \pm \tan^{-1} y; \quad \textit{and of} \quad 2\tan^{-1} x.$$

Draw two figures putting $\tan^{-1} x = \theta$; $\tan^{-1} y = \phi$.



$$\tan^{-1} x + \tan^{-1} y = \theta + \phi$$

$= \tan^{-1}\{\tan(\theta+\phi)\}$

$= \tan^{-1}\left\{\dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\right\}$

$= \tan^{-1}\dfrac{x+y}{1-xy}$

$= \sin^{-1}\{\sin(\theta+\phi)\}$; etc.

$= \sin^{-1}\{\sin\theta\cos\phi + \cos\theta\sin\phi\}$

$= \sin^{-1}\left\{\dfrac{x}{\sqrt{1+x^2}}\cdot\dfrac{1}{\sqrt{1+y^2}} + \dfrac{y}{\sqrt{1+y^2}}\cdot\dfrac{1}{\sqrt{1+x^2}}\right\}$;

obviously $\quad \tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$.

Again,

$2\tan^{-1} x = 2\theta = \tan^{-1}(\tan 2\theta)$

$= \tan^{-1}\left\{\dfrac{2\tan\theta}{1-\tan^2\theta}\right\}$

$= \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$= \cos^{-1}(\cos 2\theta)$; etc.

$= \cos^{-1}(2\cos^2\theta - 1)$

$= \cos^{-1}\left(\dfrac{2}{1+x^2} - 1\right)$

$= \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

The values of $2\sin^{-1} x$, $2\cos^{-1} x$, $2\tan^{-1} x$ might obviously have been obtained from those of $\sin^{-1} x + \sin^{-1} y$, etc., by putting $x = y$.

NUMERICAL EXAMPLES.

195. Ex. 1. Prove that

$$\cos^{-1}\tfrac{16}{65} - \cos^{-1}\tfrac{12}{13} = \sin^{-1}\tfrac{4}{5}.$$

45. The adjacent sides of a parallelogram measure a centimetres and b centimetres, and contain an angle β. Prove that the angle at which the diagonals intersect is given by

$$\cos\theta = \pm\frac{a^2 - b^2}{\sqrt{a^4 - 2a^2b^2\cos 2\beta + b^4}}.$$

46. Prove that

$$\operatorname{cosec}^6 A - 1 = \cot^2 A(\cot^4 A + 3\cot^2 A + 3).$$

47. In any triangle, prove that

$$c^2 - 2ac\cos\left(B + \frac{\pi}{3}\right) = b^2 - 2ab\cos\left(C + \frac{\pi}{3}\right).$$

48. Prove that

$$\tan^2\theta + \tan^2\left(\theta + \frac{\pi}{3}\right) + \tan^2\left(\theta + \frac{2\pi}{3}\right) = 9\tan^2 3\theta + 6.$$

49. Prove that

$$3\tan\alpha - 2\cot\alpha = \operatorname{cosec} 2\alpha - 5\cot 2\alpha.$$

50. If C be the mid-point of an arc AB of a circle, centre O, and if OC cut the chord AB in D, show that the area of the segment ACB of the circle is $R^2(\theta - \sin\theta\cos\theta)$ where $\operatorname{vers}\theta = \dfrac{CD}{R}$, and R is the radius of the circle.

51. In any triangle, prove that

$$(a + b - 2c)^2\sec^2\frac{C}{2} + (a - b)^2\operatorname{cosec}^2\frac{C}{2}$$
$$= (b + c - 2a)^2\sec^2\frac{A}{2} + (b - c)^2\operatorname{cosec}^2\frac{A}{2}$$
$$= (c + a - 2b)^2\sec^2\frac{B}{2} + (c - a)^2\operatorname{cosec}^2\frac{B}{2} = 16(R^2 - 2Rr).$$

52. Solve the equation

$$\sec 4\theta - \sec 2\theta = 2.$$

53. If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in A.P., prove that

$$4(1 - \cos\theta)(1 - \cos\phi) = \cos\theta + \cos\phi.$$

54. If

$$x - y = y - z = \mathrm{A}$$

and

$$\sin x - \sin y = \sin y - \sin z + k \sin y,$$

prove that

$$k = 2(\cos \mathrm{A} - 1).$$

55. The roof of a barn is in the shape of two similar and equal rectangles inclined at an angle β to the horizon. A person standing opposite one of the side walls at a distance b from it, finds that his eye is in the plane of the roof on that side; when he increases his distance from the wall by c, he finds that the elevation of the top of the roof is γ. Prove that the width of the barn is

$$2\left[c \cos \beta \sin \gamma \operatorname{cosec}(\beta - \gamma) - b\right].$$

56. Solve the equation

$$a \cos \theta + b \sin \theta = c.$$

57. In any triangle, prove that

$$\tan \mathrm{B} \tan \mathrm{C} + \tan \mathrm{C} \tan \mathrm{A} + \tan \mathrm{A} \tan \mathrm{B} = 1 + \sec \mathrm{A} \sec \mathrm{B} \sec \mathrm{C}.$$

58. In any triangle, show that

$$\frac{r_2 + r_3}{(s-a)\sin \mathrm{A}} = \frac{r_3 + r_1}{(s-b)\sin \mathrm{B}} = \frac{r_1 + r_2}{(s-c)\sin \mathrm{C}} = \frac{abcs}{2\Delta^2}.$$

59. Find in degrees the sum of the three acute angles,

$$\sin^{-1}\tfrac{12}{13}, \quad \cos^{-1}\tfrac{7}{25}, \quad \tan^{-1}\tfrac{467}{40}.$$

60. The sides of a square, taken in order, subtend angles $\alpha, \beta, \gamma, \delta$ at an internal point: prove that

$$\frac{1}{\cot \alpha + \cot \gamma} + \frac{1}{\cot \beta + \cot \delta} = 1.$$

61. Prove that

$$\tan 82\tfrac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2.$$

62. If $A+B+C=90^\circ$, prove that

$$\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C - \cot B \tan C - \cot C \tan B - \cot C \tan A - \cot A \tan C - \cot A \tan B - \cot B \tan A = 2.$$

63. If

$$\sin A = p \sin B, \quad \cos A = q \cos B, \quad \sin A + \cos A = r(\sin B + \cos B),$$

prove that

$$(p-r)^2(1-q^2)+(q-r)^2(1-p^2)=0.$$

64. ACBP is a quadrilateral figure such that the angle APB (2β) is bisected by the diagonal CP. If $CA=a$, $CB=b$, and the angle $ACB=\alpha$, prove that

$$CP=\frac{ab}{\sin\beta}\cdot\frac{\sin(\alpha+2\beta)}{\sqrt{a^2+b^2+2ab\cos(\alpha+2\beta)}}.$$

65. In a four-sided field ABCD, the angles subtended by BC, DC at A are respectively 60° and 30°; the angles subtended by AD, DC at B are respectively 30° and 60°; and the length of AB is 300 feet. Find the length of CD and the area of the field.

66. Prove that

$$4\cos\theta\cos(120^\circ-\theta)\cos(120^\circ+\theta)=\cos 3\theta.$$

✓ 67. Solve

(i) $a(\cos\theta-\cos 2\theta)=b(\sin\theta-\sin 2\theta)$.

✓ (ii) $\sin x+\sin 3x=\cos 2x+\cos 4x$.

68. In any triangle, prove that

$$\cos^2(A-B)+\cos^2(A-C)+2\cos(A-B)\cos(A-C)\cos A = (1+8\sin B\sin C\cos A)\sin^2 A.$$

69. Show that if $\Sigma\cos(\beta-\gamma)=-\frac{3}{2}$, then

$$\cos^3(\alpha+\theta)+\cos^3(\beta+\theta)+\cos^3(\gamma+\theta) - 3\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta)$$

vanishes whatever be the value of θ.

70. Lines are drawn within a triangle ABC through the vertices A, B, C making the same angle θ with the sides AB, BC, CA respectively. Prove that the area of the triangle formed by these lines is to the area of the given triangle as

$$(\cot\theta-\cot A-\cot B-\cot C)^2 \;:\; \operatorname{cosec}^2\theta.$$

71. A statue 30 feet high, standing on the top of a tower, subtends at a point, distant 150 feet in a horizontal line from the base of the tower, the same angle as that subtended at the same point by a man 6 feet high standing at the base; find (to $\frac{1}{10}$ of a foot) the height of the tower.

72. Prove that

$$4(\sin 24^\circ + \cos 6^\circ) = \sqrt{3} + \sqrt{15}.$$

73. If a triangle ABC be in a horizontal plane, and an object, P, vertically above A, have angles of elevation of 60°, 45°, and 30° at B, M, and C respectively, show that AP is equal to $\frac{a\sqrt{6}}{4}$, M being the middle point of BC.

74. P is a point inside a triangle ABC at distances x, y, z from the vertices A, B, C respectively; if α, β, γ be the angles subtended at P by the sides a, b, c, show that

$$\frac{ax}{\sin(\alpha - A)} = \frac{by}{\sin(\beta - B)} = \frac{cz}{\sin(\gamma - C)} = \frac{abc}{x\sin\alpha + y\sin\beta + z\sin\gamma}.$$

75. If the tangents of the angles of a triangle are in arithmetical progression, show that the squares of the sides are in the ratios

$$x^2(x^2+9) \ : \ (3+x^2)^2 \ : \ 9(1+x^2)$$

where x is the least or greatest tangent.

76. Prove that

(i) $$\operatorname{cosec}\frac{\pi}{2} + \operatorname{cosec}\frac{\pi}{4} + \operatorname{cosec}\frac{\pi}{8} = \cot\frac{\pi}{16}.$$

(ii) $$\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}\left(\operatorname{cosec}\frac{\pi}{14} - 1\right).$$

77. If $p = 1 + \sin^2\theta$ and $q = 1 + \cos^2\theta$, show that

$$2(p^3 + q^3) + 9q^2 = 27(1 + \cos^4\theta).$$

78. Solve $\cos x - \sin x = \cos a + \sin a.$

79. In any triangle, prove that

$$(b^2 - c^2)\cot^2\frac{A}{2} + (c^2 - a^2)\cot^2\frac{B}{2} + (a^2 - b^2)\cot^2\frac{C}{2}$$
$$= -\frac{1}{r^2}(a+b+c)(b-c)(c-a)(a-b).$$

80. Prove that

$$\{\cos(\sin^{-1} x)\}^2 = \{\sin(\cos^{-1} x)\}^2.$$

81. If $\tan 3A + \tan 2A = 0$, show that $\tan A$ may have any one of the values

$$0, \ \pm\sqrt{5 \pm 2\sqrt{5}}.$$

82. The distance between the centres of two wheels is a, and the sum of their radii is c, show that the length of the string which crosses between the wheels and just wraps around them is

$$2\left\{\sqrt{a^2 - c^2} + c\cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

83. A hexagon, two of whose sides are of length a, two of length b, and two of length c, is inscribed in a circle of diameter d. Prove that

$$d^3 = (a^2 + b^2 + c^2)\,d + 2abc.$$

84. In any triangle, prove that

$$a^3 \cos B \cos C + b^3 \cos C \cos A + c^3 \cos A \cos B$$
$$= abc\,(1 - 2\cos A \cos B \cos C).$$

85. Solve the equation

$$\cos 3x \cos\beta + \sin\alpha \sin\gamma = \cos(3x - \alpha)\cos(3x - \gamma).$$

✓ 86. Prove that

$$\sin 20^\circ + \sin 50^\circ + \sin 70^\circ = 4\cos 10^\circ \cos 25^\circ \cos 55^\circ.$$

✓ 87. Prove that

$$\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ = 1 + \sin^2 9^\circ + \sin^2 18^\circ.$$

88. If $A + B + C = \pi$, prove that

$$\Sigma \cos^4 A + \Sigma \sin^4 A = 2 + \cos 2A \cos 2B \cos 2C.$$

89. If

$$\sin(\alpha + \beta + \gamma) - \cos(\alpha + \beta + \gamma) + 2\sin\alpha \sin\beta \sin\gamma$$
$$+ 2\cos\alpha\cos\beta\cos\gamma = 0,$$

then either α, β, or γ is of the form $n\pi - \dfrac{\pi}{4}$.

90. If in the 'Ambiguous Case' of a triangle O_1, O_2; G_1, G_2; P_1, P_2 be respectively the two positions of the circumcentre, centroid and orthocentre, prove that

$$2O_1O_2 = 3G_1G_2 \operatorname{cosec} A = P_1P_2 \sec A,$$

A being the given angle.

91. In a triangle which has $\Sigma \cot A < 2$, show that the least angle $> \cot^{-1} \frac{4}{3}$ and the greatest $< 90°$.

92. DEF is a triangle similar to ABC, and DE is at right angles to BC, while the vertices D, E, F lie on AB, BC, CA respectively. Prove that if a, b, c, are the sides of ABC the circum-radius of DEF is

$$\frac{a^2bc^2}{2a^2c^2 + b^2c^2 + a^2b^2 - b^4}.$$

93. Eliminate ϕ and ϕ' from

$$r = \frac{ab \cos(\theta - \phi)}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} = \frac{ab \cos(\theta - \phi')}{\sqrt{a^2 \sin^2 \phi' + b^2 \cos^2 \phi'}},$$

and $$\tan \phi \tan \phi' = -\frac{b^2}{a^2},$$

and show that $$2r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta.$$

94. In any triangle, prove that

$$a^2 \cos^2 A + b^2 \cos^2 B + c^2 \cos^2 C + 2bc \cos 2A \cos B \cos C$$
$$+ 2ca \cos 2B \cos C \cos A + 2ab \cos 2C \cos A \cos B = 0.$$

95. If $A + B + C + D = 0$, prove that

$$\sin(A + B) \sin(A - B) + \sin(C + D) \sin(C - D)$$
$$= 2[\sin B \sin D \cos A \cos C - \sin A \sin C \cos B \cos D].$$

96. ABC is an equilateral triangle, whose side is a, and P any point on the circumference of the inscribed circle; show that

$$PA^2 + PB^2 + PC^2 = \tfrac{5}{4}a^2.$$

97. Prove that the perpendiculars from the vertices of a triangle on a line joining the orthocentre and circumcentre are

$$2R \cos A \sin(B - C)/\lambda, \quad 2R \cos B \sin(C - A)/\lambda,$$

$$2R \cos C \sin(A - B)/\lambda, \quad \text{where } \lambda^2 = 1 - 8 \cos A \cos B \cos C.$$

98. A straight line AD is divided into three equal parts at B and C; the angles subtended by AB, BC, CD at any point P are θ, ϕ, ψ; prove that

$$(\cot\theta+\cot\phi)(\cot\psi+\cot\phi)=4\operatorname{cosec}^2\phi.$$

99. From a point P, perpendiculars are drawn to the n sides of a regular polygon inscribed in a circle of radius c. If the sum of the squares of these perpendiculars be nh^2, show that the distance δ of the point P from the centre of the polygon is given by

$$\delta^2=2\left(h^2-c^2\cos^2\frac{\pi}{n}\right).$$

100. Prove that

$$(1+\sin\theta)(3\sin\theta+4\cos\theta+5)$$

is a perfect square.

101. The circumference of a given circle is divided into n equal parts at A, A_1, A_2,......A_{n-1}; if the distances of the points A_1, A_2,......A_{n-1} from A be denoted by a_1, a_2,......a_{n-1}, show that

$$a_1a_2+a_2a_3+\ldots\ldots+a_{n-2}a_{n-1}=2nr^2\cos\frac{\pi}{n}.$$

102. Eliminate θ between

$$\sin\theta+\sin 2\theta=a,\ \text{ and }\ \cos\theta+\cos 2\theta=b.$$

103. A, B, C are three mountain peaks and the heights of B and C are known to be h and k respectively. At the lowest peak C, it is observed that the lines CA, CB make angles α, β with a horizontal plane and that the angle between the vertical planes through CB and CA is θ. At B it is observed that the angle between the vertical planes through BA and BC is ϕ. Prove that the height of A is

$$k+(h-k)\frac{\tan\alpha\sin\phi}{\tan\beta\sin(\theta+\phi)}.$$

104. If $\alpha+\beta+\gamma+\delta=0$, prove that

$$\frac{\tan\alpha+\tan\beta+\tan\gamma+\tan\delta}{\cot\alpha+\cot\beta+\cot\gamma+\cot\delta}=\tan\alpha\tan\beta\tan\gamma\tan\delta.$$

105. If θ_1, θ_2, θ_3, θ_4 are four values of θ not differing by multiples of 2π which satisfy the equation

$$a\sin 2\theta+b\sin\theta+c=0,$$

prove that

(i) $\Sigma \sin \theta_1 = 0.$

(ii) $4 \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 (\Sigma \sin \theta_1 \sin \theta_2 + 1)$
$$= (\Sigma \sin \theta_1 \sin \theta_2 \sin \theta_3)^2.$$

106. Prove that if the angle A of a triangle ABC is increased by x, whilst b, c are unaltered, the angle B will be increased by y, where

$$\tan y = \frac{2b \sin \frac{x}{2} \left\{c \cos\left(A + \frac{x}{2}\right) - b \cos \frac{x}{2}\right\}}{c^2 - 2bc \cos \frac{x}{2} \cos\left(A + \frac{x}{2}\right) + b^2 \cos x}.$$

107. If $$\tan(\phi - \theta) = \frac{k^2 \sin 2\phi}{1 + k^2 \cos 2\phi}$$

and $$\tan\left(\frac{\pi}{4} - \phi\right) = \sin(\theta - \alpha) \operatorname{cosec}(\theta + \alpha),$$

prove that $$\tan \alpha = \frac{1 - k^2}{1 + k^2} \tan^2 \phi.$$

108. In any triangle, prove that

$$b^2 \cos 2B + c^2 \cos 2C + 2bc \cos(B - C) = a^2 \cos 2(B - C).$$

109. Show that, if the medians BE and CF of a triangle meet at G,

$$\tan BGC = \frac{12\Delta}{b^2 + c^2 - 5a^2}.$$

110. If $\cos(A + B + C) + \cos(B + C - A)$
$$+ \cos(C + A - B) + \cos(A + B - C) = 0,$$
show that one of the angles A, B, C must be an odd multiple of a right angle.

111. Prove

$$2 \tan^{-1} \sqrt{\frac{a - b}{a + b}} \tan \frac{x}{2} = \cos^{-1} \frac{b + a \cos x}{a + b \cos x}.$$

112. If $\cos^2 2\theta + \cos^2 2\theta + \mu^2 \cos 2\theta = \mu^2$, show that

$$\mu \tan^2 \theta + \tan^2 \theta + \mu \tan \theta = 1.$$

113. If $A + B + C = 90°$, then

$$\frac{\cos A + \sin B + \sin C}{\sin A + \cos B + \sin C} = \frac{1 - \tan \frac{A}{2}}{1 - \tan \frac{B}{2}}.$$

114. If $\cos \phi - \cos \theta = m,$

and $\sin \phi - \sin \theta = n,$

show that $\operatorname{cosec} (\theta + \phi) = -\frac{m^2 + n^2}{2mn}.$

115. If $2 \cos \theta = x + \frac{1}{x},$

show that $2 \cos 3\theta = x^3 + \frac{1}{x^3}.$

116. If the bisectors of the angles A, B, C of a triangle ABC meet the opposite sides in D, E, F; prove that

$$\frac{4 \text{ (area of ABC)} \times \text{(area of DEF)}}{\text{AD . BE . CF}}$$

$$= \text{radius of the circle inscribed in ABC.}$$

117. In a triangle ABC, D is a point in BC such that BD = 2CD, show that

$$\text{AD} = \tfrac{1}{3} \sqrt{6b^2 + 3c^2 - 2a^2}.$$

118. The sides of a triangle are in Arithmetical Progression and its area is four-fifths that of an equilateral triangle of the same perimeter; show that the sides of the triangle are as

$$7 : 10 : 13.$$

119. If a straight line of length p bisect the angle A of a triangle ABC and divide the base into two parts of lengths m and n, prove that

$$p^2 = bc - mn.$$

120. Show that

$$\tan^{-1} \frac{2a - b}{b\sqrt{3}} + \tan^{-1} \frac{2b - a}{a\sqrt{3}} = \frac{\pi}{3}.$$

121. Solve

$$\frac{(66\cdot 66)^{\frac{1}{2}} \sin^2 33^\circ \sqrt{\cos 33^\circ 7'}}{(\cdot 0033)^4\, x^{\frac{1}{3}}} = \tan^5 57^\circ.$$

122. If O is the centre of the circle described round an acute-angled triangle and AO is produced to meet BC in D, show that

$$\text{OD} = \frac{\text{R} \cos \text{A}}{\cos (\text{B} - \text{C})}.$$

123. If the inscribed circle of a triangle ABC touch the sides BC, CA, AB in D, E, F, prove that $\tan \text{ADB} = \dfrac{2r_1}{b - c}$, where r_1 is the radius of the escribed circle which touches BC.

124. Show that the radius of the circle which touches the sides AB, AC of the triangle ABC and also touches the inscribed circle is

$$r \frac{1 - \sin \frac{\text{A}}{2}}{1 + \sin \frac{\text{A}}{2}}.$$

125. If in the ambiguous case the area of the larger triangle is double that of the smaller, show that the tangent of one of the angles at the base is three times that of the other.

126. Solve

$$(\sin 8^\circ + \cos 8^\circ)^{2x} = 2 \sin 16^\circ (\tan 32^\circ)^x.$$

127. Deduce from De Moivre's Theorem

$$\tan n\theta = \frac{n \tan \theta - \dfrac{n(n-1)(n-2)}{\lfloor 3} \tan^3 \theta + \ldots}{1 - \dfrac{n(n-1)}{\lfloor 2} \tan^2 \theta + \dfrac{n(n-1)(n-2)(n-3)}{\lfloor 4} \tan^4 \theta - \ldots}.$$

128. If $\qquad \tan [\log (a + ib)] = x + iy,$

prove that $\qquad 2x = (1 - x^2 - y^2) \tan [\log (a^2 + b^2)].$

129. Prove that

$$\log_e \sqrt{8} = 1 + \frac{1}{3 \cdot 9} + \frac{1}{5 \cdot 9^2} + \frac{1}{7 \cdot 9^3} + \ldots$$

130. If
$$y=\frac{x}{\underline{|1}}-\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}-\frac{x^4}{\underline{|4}}+\dots$$
show that
$$x=y+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}+\dots$$
y being numerically less than unity.

131. Prove that the length of a plane arc of small curvature is approximately
$$\frac{c-40c'+256c''}{45},$$
where $c=$ the chord of the arc, $c'=$ the chord of half the arc and $c''=$ the chord of quarter of the arc.

132. Prove that
$$\sec^2\frac{\pi}{9}+\sec^2\frac{3\pi}{9}+\sec^2\frac{5\pi}{9}+\sec^2\frac{7\pi}{9}=40.$$

133. Draw on squared paper a graph of $\tan 10x-2\tan 9x+1$ for values of x between 0° and 9°, and thus show that the expression vanishes when $x=5^\circ\cdot 9$.

134. Prove that the eliminant of
$$\frac{1}{a^2}=\frac{\cos^2\theta}{t^2}+\frac{\sin^2\theta}{t'^2};\quad \frac{1}{b^2}=\frac{\cos^2\phi}{t^2}+\frac{\sin^2\phi}{t'^2};\quad t\tan\theta\tan\phi=t',$$
is
$$a^2b^2-t^2t'^2=0.$$

135. Prove that
$$\log_e 5-\log_e 4=\frac{1}{5}+\frac{1}{2\,.\,5^2}+\frac{1}{3\,.\,5^3}+\frac{1}{4\,.\,5^4}+\dots.$$

APPENDIX I.

SEVEN FIGURE LOGARITHMS.

233. For some purposes it may be necessary to obtain a more accurate result than is possible with 4 figure logarithms.

When the logarithm of a number between 1 and 100,000 is required, the value may be written down at once from the Tables.

To obtain the *mantissa*, we look for the *first four* significant figures in the first column and passing along the row containing these, take the number in that particular column headed by the *fifth* figure; this gives the last 4 digits of the mantissa, the first 3 digits being obtained from the column headed by 0.

A bar placed over the last 4 digits has the same significance as in Art. 67, and indicates that the first 3 digits are obtained from a succeeding instead of a preceding line.

Ex. 1. Find the logarithm of [illegible].

We firstly look out the row containing [illegible] in the first column and in this row select the number headed by the fifth figure 3. This gives for the last 4 figures [illegible], and the first 3 are [illegible]. Since there are 2 figures to the left of the decimal point in the original number, it follows that the characteristic is 1,

$\therefore$ log [illegible] = [illegible].

No.	0	1	2	3	4	5	6	7	8	9	Diff.
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	

234. If the number whose logarithm is required contains more than 5 figures, we have to make use of the *Rule of Proportional Parts*, and the column of Differences on the right of the Table becomes an essential feature. This rule is that for small differences, *the increase in the logarithm of a number is proportional to the increase of the number.*

No.	0	1	2	3	4	5	6	7	8	9	Diff. 93	
4671	6694099	4192	4285	4378	4471	4564	4656	4749	4842	4935	1	9
											2	19
72	5028	5121	5214	5307	5400	5493	5586	5679	5772	5865	3	28
											4	37
73	5958	6051	6144	6237	6330	6423	6516	6608	6701	6794	5	47
											6	56
74	6887	6980	7073	7166	7259	7352	7445	7537	7630	7723	7	65
											8	74
75	7816	7909	8002	8095	8188	8281	8373	8466	8559	8652	9	84

From the table given above

$$\log 46718 = 4{\cdot}6694842$$
$$\log 46717 = 4{\cdot}6694749,$$
$$\therefore \text{ difference for } 1 = \ {\cdot}0000093.$$

The above rule gives

$$\text{diff. for } {\cdot}1 = \tfrac{1}{10} \text{ diff. for } 1 = {\cdot}0000009 \text{ (correct to 7 places)},$$
$$\text{,,} \quad {\cdot}2 = \tfrac{2}{10} \quad \text{,,} \quad = {\cdot}0000019$$
$$\text{,,} \quad {\cdot}3 = \tfrac{3}{10} \quad \text{,,} \quad = {\cdot}0000028 \text{ etc.}$$

It will be seen that these terminal figures 9, 19, 28 etc. are the same as the figures in the Difference Column, which may therefore be used in future instead of those obtained from the above calculations.

Ex. 2. Find log 4673·8723.

$$\begin{array}{lr|l} \log 4673{\cdot}8 & =3{\cdot}6696701 & \\ \textit{diff. for } 7 & 65 & \\ \qquad 2 & 1 & 9 \\ \qquad 3 & & 28, \end{array}$$
$$\therefore \log 4673{\cdot}8723 = 3{\cdot}6696768.$$

Ex. 3. Find x, given $\log x = 3{\cdot}6697402$.

$$\begin{array}{lr} & \log x = 3{\cdot}6697402 \\ \log 4674{\cdot}5 & = \underline{3{\cdot}6697352} \\ & 50 \\ \textit{diff. for } 5 & \underline{47} \\ & 30 \\ 3 & \underline{28} \end{array}$$
$$\therefore x = 4674{\cdot}553 = 4{\cdot}674553 \times 10^3.$$

[We firstly find from the Tables the mantissa next below that given, ·6697352, and noticing that the next mantissa is ·6697445 and the difference between these ·0000093, select the difference column headed by 93.]

Ex. 4. Find the value of $(\cdot002489775)^{\frac{1}{4}}$.

$$\log x = \tfrac{1}{4} \log \cdot002489775$$

$$\begin{array}{lr|l} \log \cdot0024897 & \bar{3}\cdot3961470 & \\ \textit{diff. for } 7 & 123 & \\ 5 & 8 & 8, \end{array}$$

$\therefore \log \cdot002489775 = \bar{3}\cdot3961602,$

$\therefore \log x = \tfrac{1}{4}(\bar{3}\cdot3961602)$

$= \bar{1}\cdot3490401$

$$\begin{array}{lr} \log \cdot22337 & \bar{1}\cdot3490248 \\ & 153 \\ \textit{diff. for } 7 & 137 \\ & 160 \\ 8 & 156 \end{array}$$

$\therefore x = \cdot2233778 = 2\cdot233778 \times 10^{-1}.$

Ex. 5. Find the value of

$$\left[\frac{(\cdot02587)^3 \times \sqrt{507\cdot02}}{83\cdot97 \times \cdot52}\right]^{\frac{1}{5}}.$$

$\log x = \tfrac{1}{5}[3 \log \cdot02587 + \tfrac{1}{2} \log 507\cdot02 - \log 83\cdot97 - \log \cdot52],$

$$\begin{array}{lr} 3 \log \cdot02587 = & \bar{5}\cdot2383892 \\ \tfrac{1}{2} \log 507\cdot02 = & 1\cdot2628976 \\ & \bar{4}\cdot5012868 \\ & 1\cdot6401275 \\ & 5)\bar{6}\cdot8611593 \end{array} \qquad \begin{array}{lr} \log 83\cdot97 = & 1\cdot9241242 \\ \log \cdot52 = & \bar{1}\cdot7160033 \\ \hline & 1\cdot6401275 \end{array}$$

$\therefore \log x = \bar{2}\cdot9722319$

$$\begin{array}{lr} \log \cdot093806 = & \bar{2}\cdot9722278 \\ & 41 \\ \textit{diff. for } 9 & 41 \end{array}$$

$\therefore x = 9\cdot38069 \times 10^{-2}.$

Trigonometrical Ratios.

235. In 7 figure tables the sines and cosines are given for all angles between 0° and 45° at intervals of 1 minute, difference columns being provided for calculating the seconds by means of the Rule of Proportional Parts.

Ratios of angles between 45° and 90° can be found by reading upwards from the bottom of the page.

Ex. 1. Find sin 29° 1′ 13″.

From the tables, $\sin 29°\ 1' = \cdot 4850640$

diff. for $60'' = 2544$

$\therefore$ *increase* for $13'' = 551$

$\therefore \sin 29°\ 1'\ 13'' = \cdot 4851191.$

$$\begin{array}{r} 2544 \\ 13 \\ \hline 2544 \\ 7632 \\ 60)\overline{33072} \\ \hline 551 \end{array}$$

Ex. 2. Find cos 29° 4′ 34″.

From the tables, $\cos 29°\ 4' = \cdot 8740550$

diff. for $60'' = 1413$

$\therefore$ *decrease* for $34'' = 801$

$\therefore \cos 29°\ 4'\ 34'' = \cdot 8739749.$

$$\begin{array}{r} 1413 \\ 17 \\ \hline 1413 \\ 9891 \\ 30)\overline{24021} \\ \hline 801 \end{array}$$

Ex. 3. Find the angle whose cosine is ·8741742.

From tables, $\cos 29°\ 3' = \cdot 8741963$

$\cos x = \cdot 8741742$

$\therefore$ diff. $= 221$

Now diff. for $60'' = 1413$

$\therefore$ req. no. of seconds $= \dfrac{60 \times 221}{1413}$

$= 9$ (nearly)

$\therefore x = 29°\ 3'\ 9''$ (adding, since $\cos x < \cos 29°\ 3'$, $\therefore\ x > 29°\ 3'$).

$$\begin{array}{r} 9{\cdot}3 \\ 1413)\overline{13260} \\ 12717 \\ \hline 5430 \end{array}$$

NATURAL SINES, COSINES, ETC.

29 Deg.

′	Sine	Diff.	Covers.	Chord	Co-Chord	Vers.	Diff.	Cosine	′
0	4848096	2544	5151904	5007600	1·0150768	1253803	1411	8746197	60
1	4850640	2544	5149360	5010116	1·0148261	1255214	1411	8744786	59
2	4853184	2543	5146816	5013232	1·0145754	1256625	1412	8743375	58
3	4855727	2543	5144273	5016048	1·0143247	1258037	1413	8741963	57
4	4858270	2542	5141730	5018964	1·0140740	1259450	1413	8740550	56
5	4860812		5139188	5021680	1·0138233	1260863		8739137	55
60	5000000	2519	5000000	5176380	1·0000000	1339746	1451	8660254	0
′	Cosine	Diff.	Vers.	Co-Chord	Chord	Covers.	Diff.	Sine	′

60 Deg.

236. In the case of tangents, cotangents, secants and cosecants, all values are given between 0° and 90° at intervals of 1 minute.

Ex. 4. Find tan 49° 1′ 13″.

From the tables, tan 49° 1′ = 1·1510445

diff. for 60″ = 6765

∴ *increase* for 13″ = 1466

∴ tan 49° 1′ 13″ = 1·1511911.

$$\begin{array}{r} 6765 \\ 13 \\ \hline 6765 \\ 20295 \\ 60\overline{)87945} \\ \hline 1466 \end{array}$$

Ex. 5. Find cot 34° 58′ 17″.

From the tables, cot 34° 58′ = 1·4299178

diff. for 60″ = 8852

∴ *decrease* for 17″ = 2508

∴ cot 34° 58′ 17″ = 1·4296670.

$$\begin{array}{r} 8852 \\ 17 \\ \hline 8852 \\ 61964 \\ 60\overline{)150484} \\ \hline 2508 \end{array}$$

NATURAL TANGENTS.

′	49°	50°	51°	52°	53°	54°	55°	′
0	1·1503684	1·1917536	1·2348972	1·2799416	1·3270448	1·3763810	1·4281480	60
1	1·1510445	1·1924579	1·2356319	1·2807094	1·3278483	1·3772242	1·4290320	59
2	1·1517210	1·1931626	1·2363672	1·2814776	1·3286524	1·3780672	1·4299178	58
60	1·1917536	1·2348972	1·2799416	1·3270448	1·3763810	1·4281480	1·4825610	0
′	40°	39°	38°	37°	36°	35°	34°	′

NATURAL COTANGENTS.

Logarithmic Sines, Cosines, etc.

237. These values are given for all angles between 0° and 90° at intervals of 1 minute, difference columns being provided for the seconds, and the Rule of Proportional Parts again being used.

Ex. 1. Find $L\sec 33°\,1'\,19''$.

From the tables, $L\sec 33°\,1' = 10{\cdot}0764907$

diff. for $60'' = 821$

$\therefore$ *increase* for $19'' = 260$

$\therefore L\sec 33°\,1'\,19'' = 10{\cdot}0765167.$

$$\begin{array}{r} 821 \\ \underline{19} \\ 821 \\ 7389 \\ 60\overline{)15599} \\ 260 \end{array}$$

Ex. 2. Find x, given that $L\operatorname{cosec} x = 10{\cdot}2636425$.

From the tables, $L\operatorname{cosec} 33°\,1' = 10{\cdot}2636968$

$\therefore$ diff. $= 543$

Now diff. for $60'' = 1944$

$\therefore$ req. no. of seconds $= \dfrac{60 \times 543}{1944}$

$= 17$ (nearly)

$$\begin{array}{r} 16{\cdot}7 \\ 1944\overline{)32580} \\ \underline{1944} \\ 13140 \\ \underline{11664} \\ 14760 \end{array}$$

$\therefore x = 33°\,1'\,17''$ (adding, since $L\operatorname{cosec} x < L\operatorname{cosec} 33°\,1'$, $\therefore x > 33°\,1'$).

Logarithmic Sines, etc.

33 Deg.

′	Sine	Diff.	Cosec.	Tang.	Diff.	Cotang.	Secant	D.	Cosine	′
0	9·7361088		10·2638912	9·8125174		10·1874826	10·0764086		9·9235914	60
		1944			2765			821		
1	9·7363032		10·2636968	9·8127939		10·1872061	10·0764907		9·9235093	59
		1944			2765			821		
2	9·7364976		10·2635024	9·8130704		10·1869296	10·0765728		9·9234272	58
		1942			2764			822		
′	Cosine	Diff.	Secant	Cotang.	Diff.	Tang.	Cosec	D.	Sine	′

56 Deg.

EXAMPLES XLIX.

Find the value of

1. $1278{\cdot}4 \times 9276{\cdot}4 \times {\cdot}80051$.

2. ${\cdot}005271 \times 7{\cdot}329 \times {\cdot}00082795$.

3. $827{\cdot}932 \times 51{\cdot}82 \times {\cdot}0079856$.

4. $\dfrac{87{\cdot}563 \times {\cdot}002897}{12598{\cdot}22}$.

5. $\dfrac{457{\cdot}082 \times {\cdot}002987}{421 \times {\cdot}079825}$.

6. $\dfrac{82957{\cdot}9 \times {\cdot}02981 \times {\cdot}72456}{{\cdot}00052897 \times 82476}$.

7. $\left[\dfrac{52{\cdot}478 \times {\cdot}002497}{\frac{1}{3}\sqrt[5]{{\cdot}0029875}}\right]^3$.

8. $\left[\dfrac{85{\cdot}9781 \times {\cdot}002478 \times \frac{1}{3}({\cdot}8275)}{\sqrt[3]{{\cdot}0893476}}\right]^{\frac{1}{4}}$.

9. $1729{\cdot}5 \sin 18^\circ 17' \times \cos 19^\circ 18'$.

10. ${\cdot}0025879 \tan 42^\circ 15' \times \sec 69^\circ 14'$.

11. $\sin 18^\circ 14' 57'' \times \tan 51^\circ 20' 20''$.

12. $({\cdot}0876)^3 \operatorname{cosec} 55^\circ 17' 16''$.

13. $13{\cdot}8297 \times \sqrt[3]{82{\cdot}0092} \cos 47^\circ 15' 10''$.

14. $\frac{1}{5} \times {\cdot}0008259 \times \sqrt[5]{825{\cdot}6} \cot 18^\circ 14' 50''$.

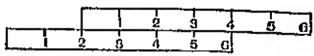
15. $\dfrac{{\cdot}02987 \tan 16^\circ 15' 40''}{\sqrt[3]{5298{\cdot}75} \operatorname{cosec} 18^\circ 17' 20''}$.

T. A. Ramalingam

APPENDIX II.

THE SLIDE RULE.

238. One method of adding together lengths is by the use of two rules placed side by side. For instance, if we wished to add 2 and 1, 2 and 2, 2 and 3 etc. we should place them as shown in the diagram, one rule overlapping the other to the extent of 2 divisions; underneath the **1** of the top rule we find the result of 2 + **1** *i.e.* 3; underneath the **2** of the top rule we find the result of 2 + **2** *i.e.* 4; underneath the **3** of the top rule we find the result of 2 + **3** *i.e.* 5, and so on.



In the same way we can subtract. If we wish, for example, to take 3 from 5, we move the rules until the 3 of the top rule coincides with the 5 of the lower one; the result of the subtraction, viz. 2, is then seen under the left-hand end of the top rule.

239. *The Slide Rule* is an instrument so graduated that we can perform multiplication and division just as easily as addition and subtraction with ordinary rules. In order to understand the principle on which it works, we merely have to remember that

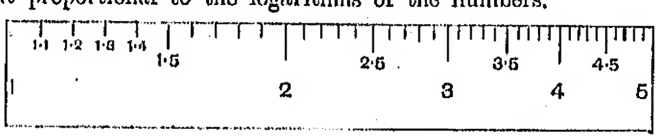
$$\log abc = \log a + \log b + \log c$$

and

$$\log \frac{a}{b} = \log a - \log b,$$

i.e. in dealing with logarithms, multiplication is replaced by addition and division by subtraction.

240. Let two rules be graduated with unequal divisions so that the distances of any two graduations from the end of the rule are not proportional to the numbers on those graduations, but proportional to the logarithms of the numbers.



The distance from 1 to 3 is not twice the distance from 1 to 2 but

$$\frac{\text{distance from 1 to 3}}{\text{distance from 1 to 2}} = \frac{\log 3}{\log 2} = \frac{\cdot 4771}{\cdot 3010}.$$

Since

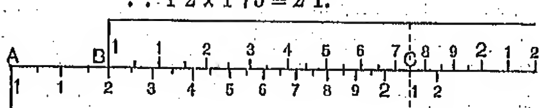
$$\begin{aligned}
\log 1 &= 0 \\
\log 2 &= \cdot 3010 \\
\log 3 &= \cdot 4771 \\
\log 4 &= \cdot 6021 \\
\log 5 &= \cdot 6990 \\
\log 6 &= \cdot 7782 \\
\log 7 &= \cdot 8451 \\
\log 8 &= \cdot 9031 \\
\log 9 &= \cdot 9542 \\
\log 10 &= 1\cdot 0000
\end{aligned}$$

it follows that the distances of the graduations 1, 2, 3 ...... 10 from the left-hand end of the rule are proportional to the numbers in the 2nd column, so that 1 is placed at the left-hand end and not 0.

Intermediate graduations are obtained by a similar process.

241. Suppose we now wish to use two such rules in order to find the value of $1\cdot 2 \times 1\cdot 75$. One of them is moved until the graduation 1—called the *Index*—is over $1\cdot 2$ of the lower rule; then looking under $1\cdot 75$ of the upper rule we find the product $2\cdot 1$ on the lower rule. The reason for this is that

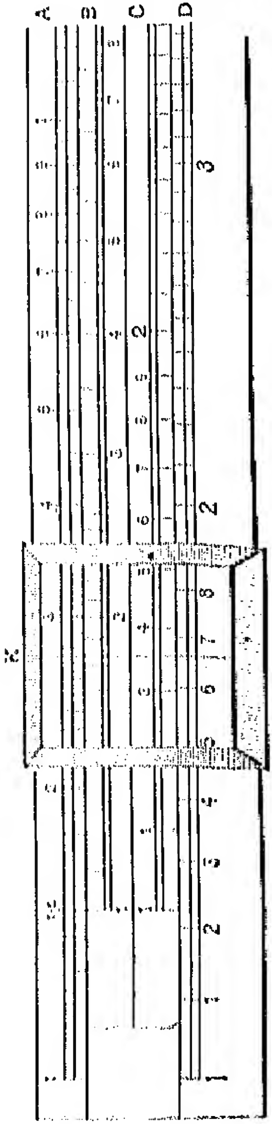
$$\begin{aligned}
\log 1\cdot 2 &= \mathrm{AB} \\
\log 1\cdot 75 &= \mathrm{BC}, \\
\therefore \log(1\cdot 2 \times 1\cdot 75) &= \log 1\cdot 2 + \log 1\cdot 75 \\
&= \mathrm{AB} + \mathrm{BC} \\
&= \mathrm{AC} \\
&= \log 2\cdot 1; \\
\therefore 1\cdot 2 \times 1\cdot 75 &= 2\cdot 1.
\end{aligned}$$



Similarly if we wish to find the value of $\frac{2\cdot 1}{1\cdot 75}$, the top rule is moved until $1\cdot 75$ on it coincides with $2\cdot 1$ on the lower rule, the

quotient 1·2 is then read off on the lower rule immediately under the index of the top rule.

242. One extremity of a Slide Rule with some of the graduations marked is shown in the diagram. It will be noticed that there are four scales; A and D being on the Rule and B and C on the Slide. Moreover A and B are graduated in the same way, and C and D in the same way, the distance between any two numbers on C or D being twice as great as that between the corresponding numbers on A or B. *The Cursor*, K, is a rectangular frame with a glass front on which is engraved a black line at right angles to the length of the Rule; the frame is made to slide in grooves.



243. *Multiplication.*

Ex. 1. Find the value of 1·96 × 1·74.

Place the index (the 1) of the C scale over 1·96 on the D scale.

The product 3·41 is then read off on the D scale under 1·74 on the C scale.

Ex. 2. Find the value of 23 × 55.

Placing the left-hand index of the C scale over 23 on the D scale we find that 55 on the C scale is off the rule. In a case like this we use the right hand index of the C scale and place it over 23 on the D scale, then under 55 on the C scale the product 1265 is read off on the D scale.

The beginner might imagine, by looking at the rule, that the last product should be 12·65; it is therefore *very important to find the position of the decimal point.* This is best done by approximating; in Ex. 2, the product is approximately 2 × 5 × 100 and therefore the answer must be 1265 and not 12·65. Rules will however be given hereafter in Art. 252.

In working examples on continued multiplication, the Cursor is of great use,

Ex. 3. Find the value of $1{\cdot}2 \times 1{\cdot}8 \times 2{\cdot}3$.

Place the index of C over 1·2 on D, then move the cursor till it is over 1·8 on C, the product of these two numbers is then under the cursor. Without reading off this product, again move C until its index is under the cursor and then the final product 4·97 is read off on D under 2·3 on C.

Ex. 4. Find the value of $2{\cdot}1 \times 3{\cdot}9 \times 2{\cdot}4$.

Place the index of C over 2·1 on D, then move the cursor till it is over 3·9 on C; the partial product is on D under the cursor.

If now, as in the last example, we again move C until its left-hand index is under the cursor, we find that 2·4 on C is *off the rule.* All that we have to do in a case of this sort is to move C so that its right-hand index is under the cursor, then the final product 19·66 is read off on D under 2·4 on C.

[The product is approx. $2 \times 4 \times 2 = 16$ and therefore the decimal point is as given.]

244. *Division.*

Ex. Divide 2·1 by 1·7.

Place 1·7 on the C scale over 2·1 on the D scale, the quotient 1·235 is then read off on the D scale under the left-hand index of C.

If the left-hand index is off the rule, the right-hand index of C is used instead.

245. *Proportion.*

To find one term of a proportion given the other three.

Find x, if $$1{\cdot}72 : 8{\cdot}7 = x : 3{\cdot}49.$$

Here obviously $$x = \frac{1{\cdot}72 \times 3{\cdot}49}{8{\cdot}7}$$

Divide 1·72 by 8·7 by placing 8·7 on the C scale over 1·72 on the D scale, then move the cursor so that it is over the right-hand index (the left-hand index being off the rule) of the C scale, *i.e.* over the quotient on the D scale.

We now have to multiply by 3·49 and do this by moving the C scale so that its left-hand index is under the cursor and the final value of x is read off on D under 3·49 on C.

It is found to be ·69.

$$\left[\text{The value of } x \text{ is approx. } \frac{2 \times 3}{9} = \frac{2}{3} = {\cdot}66.\right]$$

246. *Combined Multiplication and Division.*

Ex. Find the value of $\dfrac{243 \times 172 \times 76}{342 \times 259 \times 871}$.

Divide 243 by 342 by placing 342 on the C scale over 243 on the D scale; the quotient is on the D scale under the right-hand index of the C scale.

Multiply by 172 by **moving the cursor** to 172 on the C scale; the product is on the D scale under the cursor.

Divide by 259 by **moving the C scale** so that 259 on the C scale is under the cursor; the quotient is on the D scale under the right-hand index of the C scale.

Multiply by 76 by **moving the cursor** to 76 on the C scale; the product is on the D scale under the cursor.

Divide by 871 by **moving the C scale** so that 871 on the C scale is under the cursor; the final quotient is on the D scale under the right-hand index of the C scale.

The final result is ·0412.

$$\left[\text{The approx. value is } \frac{2 \times 2 \times 8}{3 \times 3 \times 9} = \frac{32}{81} = \cdot 4.\right]$$

Squares and Square Roots.

247. Since the distance from the index to any graduation on the C or D scale is double the distance from the index to the same graduation on the A or B scale, it follows that if any distance on the C or D scale represents $\log x$, the same distance on the A or B scale represents $2 \log x$ or $\log x^2$.

Thus above any graduation on the D scale will be found its square on the A scale.

Ex. 1. Find the square of 2·27.

Place the cursor over 2·27 on the D scale; it will then be found to be over 5·15 on the A scale.

Thus $$2{\cdot}27^2 = 5{\cdot}15.$$

[Approx. value is $2^2 = 4$.]

Ex. 2. Find the square of 178·5.

Place the cursor over 178·5 on the D scale, it will then be found to be over 31860 on the left-hand A scale.

$$\therefore\ 178{\cdot}5^2 = 31860 = 3{\cdot}186 \times 10^4.$$

[Approx. value is $180^2 = 32400$.]

248. In finding square roots, the following rules determine which scales to use.

1. If the number >1 mark off periods of two digits from the decimal point to the left, and ascertain how many digits are left in the last period marked:

If the number <1, ascertain how many significant figures there are in the first period to the right of the decimal point containing significant figures.

2. If this period contains *one* digit, use the *left-hand* A scale (since in this case the first figure of the square root cannot be greater than 3, and must therefore be found on the left-hand half of D).

3. If this period contains *two* digits, use the *right-hand* A scale.

Or, if the number be written as a multiple of a power of 10, then

i. The *left-hand* A scale is used if this power is *even*;

ii. The *right-hand* A scale is used if this power is *odd*.

e.g. $77{\cdot}5=7{\cdot}75\times10^1$. Use the right-hand A scale;
$\cdot000757=7{\cdot}57\times10^{-4}$. Use the left-hand A scale.

Ex. 3. Find the square root of 77·5.

Here there are an even number of digits in the last period marked and we therefore use the right-hand A scale.

Place the cursor over 77·5 on the A scale, and it is then over 8·8 on the D scale,

$$\therefore \sqrt{77{\cdot}5}=8{\cdot}8.$$

[Approx. value $=\sqrt{81}=9$.]

Ex. 4. Find the square root of ·000757.

Since there is one significant figure in the first period containing significant figures, the left-hand A scale is used.

Place the cursor over ·000757 on the A scale, and it will then be over ·0275 on the D scale,

$$\therefore \sqrt{\cdot000757}=\cdot0275=2{\cdot}75\times10^{-2}.$$

[Approx. value $=\sqrt{\cdot0009}=\cdot03$.]

Cubes and Cube Roots.

249. Having seen how to find the square of a number, we merely have to multiply this square by the number itself, and thus obtain the *cube*.

Ex. 1. Find the cube of 114·2.

Place the left-hand index of C opposite 1142 on D; the number on the A scale opposite this index is obviously the square of 1142.

Now multiply by 1142 again, by looking at the number on the A scale opposite 1142 on the B scale; we find 1490000 = 1·49 × 10^6.

[Approx. value = 110^3 = 1331 × 10^3 = 1·331 × 10^6.]

For alternative methods, see Art. 257, Exs. i, iii, v, ix.

250. By reversing this process we obtain *Cube Roots.* The slide must be moved until the number on the B scale under the given cube on the A scale is the same as the number on the D scale under the index on the C scale.

The following rules determine which of the scales on A and B are to be used:

1. If the number > 1, mark off periods of 3 digits from the decimal point to the left and ascertain how many digits are in the last period marked.

2. If the number < 1, ascertain the number of significant digits in the first period of 3 digits, containing significant figures, to the right of the decimal point.

3. If this period contains *one digit*, use left-hand of A and left-hand of B.

4. If this period contains *two digits*, use right-hand of A and left-hand of B.

5. If this period contains *three digits*, use right-hand of A and right-hand of B.

Or again, if the number be written as a multiple of a power of 10, then

i. If this power is a multiple of 3, use the left-hand of A and left-hand of B;

ii. If this power is 1 + a multiple of 3, use the right-hand of A and left-hand of B;

iii. If this power is 2 + a multiple of 3, use the right-hand of A and right-hand of B.

Ex. 2. Find the cube root of 39·5.

Marking the periods from the decimal point to the left, the last period contains *two digits.*

Therefore use the right-hand A scale and left-hand B scale.

The cursor is placed over 395 on the right-hand of A and the slide moved to the right until the number on the left-hand of B under the

cursor is found to be the same as the number on D un[der the]
index of C.

We thus obtain $\sqrt[3]{33\cdot5} = 3\cdot215$.

[Approx. value $= \sqrt[3]{27} = 3$.]

To find the logarithm of a number.

251. Move the slide until the index on C is [over the]
number on D, then turn the whole slide-rule ov[er and read the]
number on the middle set of graduations (reading [from right to]
left) opposite the black mark in the notch. [This is the]
mantissa.

For alternative method, see Art. 257, Ex. ix.

Ex. Find log 3.

Move the slide until the left-hand index of C is over [3 on D.]
Invert the slide-rule and ·477 is found opposite the [black mark.]

252. *Rule for determining the position of the dec[imal point in a]*
product. The number of *digits* in a product is the sum [of the number of]
the digits in the two factors, if the multiplication is [performed with]
the slide projecting to the left; while it is one less if th[e slide projects]
to the right.

If there are more than two factors, the same ru[le is suc]cessively applied. Thus the sum of the digits of [all the factors]
is obtained and 1 subtracted each time a multiplicatio[n is performed]
with the rule to the right.

N.B. If a number >1, then by the *number of digi[ts we mean the]*
number of figures to the left of the decimal point.

If a number <1 and starts with cyphers, by the nu[mber of digits]
we mean the number of cyphers coming immediately af[ter the decimal]
point.

Ex. To find the product of $2\cdot4 \times 3\cdot7 \times \cdot0059$.

Place the left-hand index of C over 2·4 on D and m[ove the cursor]
to 3·7 on C. [The slide projects to the right.] No[w multiply by]
·0059 by placing the right-hand index of C under the c[ursor, and read]
off the final product on D under ·0059 on C. [The slide [projects to the]
left.]

The final product gives the figures 524 and we hav[e to determine]
where to put the decimal point.

The number of *digits* in the original factors is 1 +[1 − 2 = 0. From]
this we have to subtract 1, since *one* multiplication was [performed with]
the slide projecting to the right.

Therefore number of *digits* in product is -1, an[d]
the product is ·0524.

253. *Rule for determining the position of the decimal point in a quotient.*

The number of *digits* in a quotient is the same as the excess of the number of digits in the dividend over the number in the divisor, if the slide is projecting to the left; while it is one more if the slide projects to the right.

Ex. Divide 5·01 by ·0322.

Place ·0322 on C over 5·01 on D. [The slide projects to the right.] The quotient 1556 is then found under the left-hand index of C. To determine the position of the decimal point, we find, by the above rule, the number of digits in the quotient to be $1-(-1)+1=3$.

Therefore quotient = 155·6.

EXAMPLES 1.

Multiply

1. 7·43 by 8·5.
2. 3·46 by 71·2.
3. ·0431 by ·00723.
4. 8·557 by ·0241.

Evaluate

5. 43·17 × 22·03 × ·718.
6. ·00295 × 72·4 × 15·8.
7. 13·21 × 18·9 × ·075.

Divide

8. ·1125 by ·015.
9. 603 by 8·4.
10. 182·9 by ·517.
11. ·308 by ·0072.

Find the value of x in the following equations:

12. $7·21 : 8·3 = 3 : x$.
13. $5·27 : x = ·021 : 4·95$.
14. $x : 61·3 = 72·41 : 80·5$.
15. $17·3 : 15·4 = x : 92·7$.

Find the value of

16. $\dfrac{19·3 \times 10·5}{7·3 \times 14·31}$.
17. $\dfrac{137·5 \times 4·09 \times 81}{772 \times 690}$.
18. $\dfrac{52·41 \times 71·43 \times 1·41}{11·37 \times 15·3 \times 16·9}$.
19. $\dfrac{10·24 \times 17·81 \times 15·21}{202 \times 16·91 \times 18·24}$.

20. Find the squares of (i) 10·5. (ii) ·0527. (iii) 18·54. (iv) 32·10. (v) ·00543.

21. Find the square roots of (i) 85·5. (ii) 1035. (iii) ·0724. (iv) ·000935. (v) 1559.

22. Find the cubes of (i) 75·9. (ii) 821·5. (iii) ·035. (iv) ·0059.

23. Find the cube roots of (i) 72·8. (ii) 824·5. (iii) 7·98. (iv) ·00582. (v) ·000785.

Sines and Tangents.

254. *To find the sine of an angle.* (i) Invert the whole slide-rule and move the scale of sines until the necessary number of degrees comes opposite the black mark; then turning the whole slide-rule over again, the required value of the sine is found on B opposite the right-hand index of A.

If the result is found on the right-hand B scale, a decimal point is put at the beginning; while if it is found on the left-hand B scale, a cipher is first placed at the beginning and then the decimal point in front of the cipher.

Ex. To find sin 30°.

Turn the slide-rule over and draw out the slide until 30 on the Sine scale is opposite the black mark. Then we find 5 opposite the right-hand index of A.

Thus sin 30° = ·5.

(ii) A second method is to take the slide right out and then put it back again with the Sine scale next to the A scale and its extremities coinciding with the extremities of the A scale. The sines of all the angles are then read off on the A scale opposite the corresponding number of degrees on the Sine scale.

Between 70° and 90° the graduations, if marked, would be extremely close together, so that only 75° and 80° are indicated. The sines of other angles between 70° and 90° may be obtained from any of the approximate rules found in books on the Slide Rule.

255. *To find the tangent of an angle.* (i) Invert the whole slide-rule and move the scale of tangents until the necessary number of degrees comes opposite the black mark; on turning the slide rule over again the value of the tangent is found on A opposite the right-hand index of B.

As in the case of the sines, a decimal point is prefixed, if the result is found on the right-hand A scale; a decimal point and a cipher if the result is on the left-hand A scale.

Turn the slide-rule over and draw out the slide until 5 on the Tangent scale is opposite the black mark; we then find 875 opposite the right-hand index of the B scale.

Therefore $\tan 5° = \cdot 0875$.

(ii) The second method is to take the slide out and re-insert it with the Tangent scale next to the A scale and the extremities coinciding.

The tangents of all the angles up to 45° are then read off on the A scale opposite the corresponding angle on the Tangent scale.

For angles between 45° and 90°, we obtain the tangents from the formula

$$\tan A = \frac{1}{\cot A} = \frac{1}{\tan(90° - A)},$$

$$e.g.\ \tan 60° = \frac{1}{\tan 30°}.$$

APPLICATIONS.

256. Ex. 1. Find the number of degrees in 2·57 radians.

$$1 \text{ radian} = 57{\cdot}3°,$$

$$\therefore\ 2{\cdot}57 \text{ radians} = 57{\cdot}3° \times 2{\cdot}57 = 147{\cdot}3°.$$

[Place the right-hand index of C opposite 57·3 on D, then under 2·57 on C we read 147·3 on D.]

Ex. 2. Find the circumference of a circle when the diameter is 12 inches.

$$\text{Circumference} = \pi d = \pi \times 12$$
$$= 37{\cdot}7 \text{ inches}.$$

[Place the left-hand index of B opposite π (specially marked) on A; then opposite 12 on B we read 377 on A.]

Ex. 3. Find the area of a circle when the diameter is 6 inches.

$$\text{Area} = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 6^2 \text{ sq. inches} = 28{\cdot}3 \text{ sq. inches}.$$

[Divide π by 4 by placing 4 on B underneath π on A (the quotient is on A over the right-hand index of B); then multiply by d^2 by observing the reading on A opposite 6 on C. We obtain 283.]

Ex. 4. Find the volume of a sphere 5·7 cms. in radius.

$$\text{Volume} = \tfrac{4}{3}\pi r^3 = 4{\cdot}189 \times 5{\cdot}7^3 \text{ cu. cms. (since } \pi = 3{\cdot}142)^*$$
$$= 776 \text{ cu. cms.}$$

[Multiply 4·189 by 5·7 by moving the slide till the left-hand index of B is under 4·189 on A, then move the cursor to 5·7 on B. Multiply this result by 5·7² by moving the slide till the right-hand index is under the cursor, and the final result is on A opposite 5·7 on C.]

Ex. 5. Find the area of a triangle, the sides being 27·5, 22·4 and 19·8 cms. respectively.

$$\begin{aligned} a &= 27{\cdot}5 \\ b &= 22{\cdot}4 \\ c &= 19{\cdot}8 \\ 2&\underline{|69{\cdot}7.} \\ \therefore\ s &= 34{\cdot}85, \end{aligned}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{34{\cdot}85 \times 7{\cdot}35 \times 12{\cdot}45 \times 15{\cdot}05} = 220 \text{ sq. cms.}$$

[Since we eventually have to take a square root, it will be convenient to work with the A and B scales.

Place the left-hand index of B on 34·85 of the left-hand A scale and the cursor on 7·35 of the left-hand B scale.

Move the left-hand index of B to the cursor, and then the cursor to 12·45 on the left-hand B scale.

Move the left-hand index of B to the cursor and the final product of the four factors is found on A opposite 15·05 on B. By a rough calculation the product contains 5 digits and is therefore 48000.

To find the square root, we place the cursor over 48000 on the left-hand A scale (since there is an odd number of digits) and find it is then over 220 on D.]

Ex. 6. Find B and C given that $b = 17{\cdot}2$, $c = 15{\cdot}4$ and the included angle $A = 38°\ 40'$.

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$$= \frac{1{\cdot}8}{32{\cdot}6}\cot 19°\ 20' = \frac{1{\cdot}8}{32{\cdot}6} \times \frac{1}{\tan 19°\ 20'} = {\cdot}157.$$

* In finding the volumes of spheres, it will in future be advisable to remember that

$$\tfrac{4}{3}\pi = 4{\cdot}189.$$

[Inverting the slide-rule and placing 19° 20′ on the Tangent scale opposite the black mark, then turning the slide-rule over, we read ·351 opposite the right-hand index of the B scale,

$$\therefore \tan\frac{B-C}{2}=\frac{1\cdot 8}{32\cdot 6}\times\frac{1}{\cdot 351}.$$

Place 32·6 on the C scale opposite 1·8 on the D scale; the quotient is then on the D scale, under the right-hand index on the C scale.

Put the cursor at this place and then move the slide until ·351 on the C scale is under the cursor; the final result ·157 is then found on the D scale under the left-hand index of C.]

To obtain $\frac{B-C}{2}$, we move the slide until the right-hand index of B is under ·157 on the right-hand A scale; turning the rule over and looking at the black mark against the Tangent scale, we find that

$$\frac{B-C}{2}=8°\,55'.$$

Now
$$\frac{B+C}{2}=90°-\frac{A}{2}=70°\,40',$$

$$\therefore\ B=79°\,35',$$

$$C=61°\,45'.$$

Ex. 7. Given that 1 inch = 2·54 centimetres, find the number of centimetres in 537 inches.

Place the right-hand index of C opposite 2·54 on D, then under 537 on C we read 1364 on D;

$$\therefore\ 537 \text{ inches}=1364 \text{ centimetres}.$$

Ex. 8. Find $3\frac{1}{2}$ % of $115\frac{1}{2}$.

$$3\tfrac{1}{2}\,\%\text{ of }115\tfrac{1}{2}=\frac{115\cdot 5\times 3\cdot 5}{100}=1\cdot 155\times 3\cdot 5=4\cdot 04.$$

[Place the left-hand index of C opposite 1·155 on D, then under 3·5 on C we read 4·04 on D.]

Ex. 9. Find the space fallen through (in vacuo) by a body in 27 seconds.

$$s=\tfrac{1}{2}gt^2=16\times 27^2\text{ ft.}=11660\text{ ft. (approx.).}$$

[Place the left-hand index of C opposite 27 on D, then opposite 16 on B we find 11660 on A.]

257. Mr A. G. Thornton, S. Mary's Street, Manchester, is now selling a new Slide Rule called the "Improved Perry Calculating Slide Rule." It has very many advantages; we

will here consider some of the special advantages attaching to the Log Log Scales which are marked on it.

Between the edge and the scale A is another scale called E, in which the markings are proportional to the logarithm taken twice of each number.

Thus the position of 10 is the zero position, for $\log\log 10 = 0$, and 10 is placed at a convenient point of the scale, then 4 is placed to the *left* of 10 at a distance proportional to $\log\log 4$ or $-\cdot 2204$; 50 is placed to the *right* of 10 at a distance proportional to $\log\log 50$ or $\cdot 2303$ and so on.

Between the other edge and the scale D is another scale called E^{-1} on which the graduations are the reciprocals of those on E, thus 4 on E corresponds with $\cdot 25$ on E^{-1}, 50 on E with $\cdot 02$ on E^{-1} and so on.

The following are the most important types of calculation, and the student who has the Rule in his hands will readily follow the method of working.

(i) Calculate x from $x = 2\cdot 31^{1\cdot 32}$.

Set B, 1 on E, 2·31 then find B, 1·32 and read off the answer E, 3·02.

Reason.
$$\log x = 1\cdot 32 \log 2\cdot 31,$$
$$\therefore \log\log x = \log 1\cdot 32 + \log\log 2\cdot 31.$$

(ii) Calculate x from $x = 2\cdot 31^{-1\cdot 32}$.

Proceed just as in (i) but opposite B, 1·32 read off the answer E^{-1}, ·33.

Reason. We really calculate as in (i) and read off the reciprocal of the answer.

(iii) Calculate x from $x = \cdot 568^{1\cdot 52}$.

Set B, 1 on E^{-1}, ·568 then find B, 1·52 and read off the answer E^{-1}, ·423.

Reason. It is not possible to take $\log\log \cdot 568$; we have therefore to use the reciprocal $\frac{1}{\cdot 568}$, the process is then the same as in (i) except that the reciprocal scale E^{-1} takes the place of E all through, thus
$$\log\log \frac{1}{x} = \log 1\cdot 52 + \log\log \frac{1}{\cdot 568}.$$

(iv) Calculate x from $x = \cdot568^{1\cdot52}$.

Set B, 1 on E^{-1}, ·568 then find B, 1·52 and read off the answer E, 2·36.

Reason. We really calculate as in (iii) and read off the reciprocal of the answer.

(v) Calculate x from $x = 2\cdot31^{\frac{1}{1\cdot32}}$.

Set B, 1·32 on E, 2·31 then find B, 1 and read off the answer E, 1·89.

Reason.
$$\log x = \frac{1}{1\cdot32}\log 2\cdot31,$$
$$\therefore \log\log x = \log\log 2\cdot31 - \log 1\cdot32.$$

(vi) Calculate x from $x = 2\cdot31^{-\frac{1}{1\cdot32}}$.

Proceed just as in (v) but opposite B, 1 read off the answer E^{-1}, ·53.

Reason. We really calculate as in (v) and read off the reciprocal of the answer.

(vii) Calculate x from $x = \cdot568^{\frac{1}{1\cdot32}}$.

Set B, 1·32 on E^{-1}, ·568 then find B, 1 and read off the answer E^{-1}, ·652.

Reason. It is not possible to take $\log\log\cdot568$; we have therefore as in (iii) to use reciprocals. Thus we proceed exactly as in (v) using the E^{-1} instead of the E scale throughout.

(viii) Calculate x from $x = \cdot568^{-\frac{1}{1\cdot32}}$.

Proceed just as in (vii) but opposite B, 1 read off the answer E, 1·54.

Reason. We really calculate as in (vii) and read off the reciprocal of the answer.

(ix) Calculate x from $x = \log_{1\cdot32} 2\cdot31$.

$$\text{i.e. Solve } 1\cdot32^x = 2\cdot31.$$

Set B, 1 on E, 1·32 then find E, 2·31 and read off the answer B, 3·01.

Reason. $\log x = \log\log 2\cdot31 - \log\log 1\cdot32.$

T. A. Natarajan. P.S.S.6.

EXAMPLES LI.

1. Find the number of degrees in 7·2 radians.

2. Calculate the number of radians in 62°.

3. Find the number of sq. centimetres in a circle of radius 5·8 cms.

4. What is the number of centimetres in the circumference of a circle of radius 7·2 cms.?

5. Find the volume of a sphere of radius 13·2 decimetres.

6. Calculate the number of degrees in 3·4 radians.

7. Obtain the circumference of a circle of diameter 7 centimetres.

8. Find the area of a circle of diameter 16 centimetres.

9. Find the number of radians in 140°.

10. If the volume of a sphere is 18500 cu. centimetres, what is the radius?

11. Find the volume of a sphere when the radius is 15·9 centimetres.

12. Calculate the radius of a circle whose area is 1000 sq. centimetres.

13. Find the area of a triangle when the sides are 15, 17·5 and 19·5 centimetres respectively.

14. Calculate the angles B and C of a triangle, given that $b=7·5$, $c=3·2$ and $A=50°$.

15. If there are 1·609 kilometres in 1 mile, how many kilometres are there in 827 miles?

16. Calculate the value of 23 tons, if 1 lb. = 2·205 kilograms.

17. Find the number of centimetres in 5 miles, if 1 ft. = 30·48 cms.

18. Find the values of the angles C and A of a triangle, if $c=18·75$, $a=14·21$ and $B=74°50'$.

19. Calculate the area of a triangle, the sides of which are 24·7, 59·8 and 62·5 centimetres respectively.

20. Given that the earth's radius is $6·371\times10^8$ centimetres, find its value in miles, when 1 foot = 30·48 cms.

21. Find the mass of the earth in tons, given that it is $6·14\times10^{27}$ grams, and that 1 lb. = 453·6 grams.

TABLES OF LOGARITHMS, SINES, ETC.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	13	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
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·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
											Minutes				
0°	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Mean Differences 1'	2'	3'	4'	5'
45°	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	6	8	10
46	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	6	8	10
47	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	6	8	10
48	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	6	8	10
49	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	6	8	9
50	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	6	7	9
51	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	5	7	9
52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	4	5	7	9
53	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	3	5	7	9
54	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	3	5	7	8
55	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	3	5	7	8
56	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	3	5	6	8
57	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	3	5	6	8
58	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	2	3	5	6	8
59	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	6	7
60	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	6	7
61	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	6	7
62	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	5	7
63	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	5	6
64	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	5	6
65	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	3	4	5	6
66	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	5	6
67	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	6
68	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
69	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
70	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
71	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
72	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	4
73	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	2	3	4
74	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	3	4
75	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	3	3	4
76	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	3
77	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	3
78	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3
79	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3
80	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
81	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
82	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
83	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	2
84	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	1
85	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	1	1	1
86	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	1	1	1
87	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	1	1
88	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0
89	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	6	10	13	17
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	10	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	10	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
											Minutes				
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1.0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1.0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1.1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1.1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1.2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1.2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1.3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1.3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1.4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1.5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1.6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1.7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1.8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1.8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1.9626	9711	9797	9883	9970	2.0057	0145	0233	0323	0413	15	29	44	58	73
64	2.0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2.1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2.2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2.3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2.4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2.6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2.7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2.9042	9208	9375	9544	9714	9887	3.0061	0237	0415	0595	29	58	87	116	145
72	3.0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3.2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3.4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3.7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4.0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	214	267
77	4.3315	3662	4015	4373	4737	5107	5483	5864	6252	6646	62	124	186	248	310
78	4.7046	7453	7867	8288	8716	9152	9594	5.0045	0504	0970	73	146	220	293	366
79	5.1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	87	175	262	350	437
80	5.6713	7297	7894	8502	9124	9758	6.0405	1066	1742	2432					
81	6.3138	3859	4596	5350	6122	6912	7720	8548	9395	7.0264					
82	7.1154	2066	3002	3962	4947	5958	6996	8062	9158	8.0285					
83	8.1443	2636	3863	5126	6427	7769	9152	9.0579	2052	3572					
84	9.5144	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20					
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					

The value of the tangent here increases so rapidly that common difference columns cannot be used.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes				
											1′	2′	3′	4′	5′
0°	Inf. Neg.	7·2419	5429	7190	8439	9408	$\overline{0}200$	$\overline{0}870$	$\overline{1}450$	$\overline{1}961$					
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	21	41	62	83	103
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	8·9403	9489	9573	9655	9736	9816	9894	9970	$\overline{0}046$	$\overline{0}120$	13	26	39	53	65
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
45°	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	4	5	6
46	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	4	5	6
47	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	5	6
48	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	6
49	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
50	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
51	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
52	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
53	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
54	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	4	5
55	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	3	4
56	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	3	3	4
57	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	2	3	4
58	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	2	2	3	4
59	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	4
60	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	4
61	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	3
62	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	3
63	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	3	3
64	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3
65	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3
66	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3
67	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1	1	2	2	3
68	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
69	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
70	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
71	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
72	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
73	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	2	2
74	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	2
75	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	2
76	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	2
77	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	1
78	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	1	1	1	1
79	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	1	1	1
80	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	1	1	1
81	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	1	1	1
82	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	1	1	1
83	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	1	1
84	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	1
85	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0
86	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0
87	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0
88	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0
89	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0	0	0	0	0
90	[illegible]														

		6′	12′	18′	24′	30′	36′	42′	48′	54′	Minutes 1′	2′	3′	4′	5′
0°	Inf. Neg.	7·2419	5429	7190	8439	9409	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$962					
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	29	58	87	116	145
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	21	41	62	83	103
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	8·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}$068	$\bar{0}$143	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	10.0000	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
46	10.0152	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
47	10.0303	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
48	10.0456	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
49	10.0608	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
50	10.0762	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
51	10.0916	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
52	10.1072	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	10	13
53	10.1229	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	11	13
54	10.1387	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	11	13
55	10.1548	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	11	14
56	10.1710	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	5	8	11	14
57	10.1875	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	8	11	14
58	10.2042	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	9	11	14
59	10.2212	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	9	12	14
60	10.2386	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	9	12	15
61	10.2562	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	9	12	15
62	10.2743	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	9	12	15
63	10.2928	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	9	13	16
64	10.3118	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	6	10	13	16
65	10.3313	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	7	10	13	17
66	10.3514	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	3	7	10	14	17
67	10.3721	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	7	11	14	18
68	10.3936	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	7	11	15	18
69	10.4158	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	8	12	15	19
70	10.4389	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	8	12	16	20
71	10.4630	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	8	13	17	21
72	10.4882	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4	9	13	18	22
73	10.5147	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5	9	14	19	23
74	10.5425	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5	10	15	20	25
75	10.5719	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5	10	16	21	26
76	10.6032	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6	11	17	22	28
77	10.6366	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6	12	18	24	30
78	10.6725	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6	13	19	26	32
79	10.7113	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7	14	21	28	35
80	10.7537	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8	16	23	31	39
81	10.8003	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9	17	26	35	43
82	10.8522	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	10	20	29	39	49
83	10.9109	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	11	23	34	45	56
84	10.9784	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	14	27	41	55	68
85	11.0580	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	16	33	49	65	82
86	11.1554	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21	41	62	83	104
87	11.2806	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	28	55	83	110	138
88	11.4569	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]					
89	11.7581	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]					

ANSWERS.

Examples I. (page 4).

1. 312′; 3598′; 7972′.
2. 43098″; 177019″; 379171″.
3. 512′; 6009′; 13298′.
4. 121308″; 519291″; 1129204″.
5. 7^g; $20^g\,50'$; $63^g\,70'$.
6. $34^g\,20'$; $70^g\,72'\,30''\cdot2469135802$; $39^g\,8'\,61''\cdot1$.
7. $1°\,10'\,15''\cdot6$; $16°\,30'\,25''\cdot2$; $50°\,35'\,52''\cdot8$.
8. $23°\,19'\,36''\cdot084$; $57°\,37'\,38''\cdot82$; $71°\,56'\,10''\cdot668$.

Examples II. (page 8).

1.	90°.	8.	20°.	15.	63°.	22.	$\frac{3\pi}{10}$.
2.	60°.	9.	120°.	16.	3°·78.	23.	$\frac{\pi}{3}$.
3.	45°.	10.	135°.	17.	$\frac{\pi}{12}$.	24.	$\frac{5\pi}{12}$.
4.	36°.	11.	$37\frac{1}{2}°$.	18.	$\frac{\pi}{10}$.	25.	$\frac{\pi}{2}$.
5.	30°.	12.	501°.	19.	$\frac{\pi}{6}$.	26.	$\frac{2\pi}{3}$.
6.	$25\frac{5}{7}°$.	13.	$94\frac{6}{11}°$.	20.	$\frac{\pi}{5}$.	27.	$\frac{3\pi}{4}$.
7.	$22\frac{1}{2}°$.	14.	12°·6.	21.	$\frac{\pi}{4}$.	28.	π.

29. ·0908; ·3063; 1·4425. 30. ·0704; ·3386; 1·3698.

31. $60°, \frac{\pi}{3}$; $90°, \frac{\pi}{2}$; $108°, \frac{3\pi}{5}$; $120°, \frac{2\pi}{3}$; $135°, \frac{3\pi}{4}$.

32. $52\frac{21}{22}°$, $32\frac{21}{22}°$ or $\frac{233}{252}$, $\frac{145}{252}$ radians.

33. $105°$ or $1\frac{5}{6}$. 34. $3436\frac{4}{11}$.

Examples III. (page 11).

1.	$31\frac{3}{7}$ cms.	2.	$7\frac{21}{22}$ cms.	3.	$\frac{1}{720}''$; $\frac{7}{88}''$.
4.	$1833\frac{1}{3}$ sq. ft.	5.	$3\frac{1}{4}$.	6.	$103\frac{1}{11}°$.
7.	$9\frac{8}{13}$ ins.	8.	$18\frac{18}{55}$ ins.	9.	$1\frac{37}{315}$ metres.
10.	3·175 cms.	11.	1537 mls.	12.	$1\frac{9}{35}$ ft.
13.	856700 mls.	14.	2165 mls.	15.	$5\frac{85}{88}$ mls.
16.	$\frac{\cdot 807}{1}$.	17.	11742 mls. per hour.		
18.	$28\frac{2}{7}$ sq. cms.	19.	7 ins.	20.	$5\frac{115}{252}$ sq. ft.
21.	12 sq. ft.	22.	$6\frac{1}{9}$ metres.	23.	$346\frac{1}{2}$ sq. in.
24.	$1\frac{429}{1200}$ sq. ft.	25.	25 sq. in.	26.	4 ft. 5·8 ins.
27.	866760 mls.	28.	$22\frac{1}{2}°$.	29.	$\frac{5}{11}$ in.

Examples IV. (page 15).

1. $\frac{4}{3}, \frac{5}{4}, \frac{5}{3}$. 2. $3, \frac{3}{2}, \frac{3}{\sqrt{13}}$.

3. $\frac{\sqrt{231}}{16}, \frac{5}{16}, \frac{5}{\sqrt{231}}, \frac{\sqrt{231}}{5}, \frac{5}{16}, \frac{\sqrt{231}}{16}$; they are equal.

4. 13 inches, $\frac{12}{5}$, $\frac{12}{5}$, $\frac{13}{5}$, $\frac{12}{13}$; they are equal.

5. 1, 1; the value is 1 for all angles.

6. $\frac{9}{41}, \frac{9}{40}, \frac{41}{40}, \frac{40}{41}$; $\tan B = \frac{\sin B}{\cos B}$.

7. $\frac{3}{5}$. 8. $\frac{7}{\sqrt{3649}}, \frac{7}{60}$.

9. $\frac{4}{5}, \frac{3}{4}, \frac{5}{7}, \frac{\sqrt{74}}{7}$.

10. $\frac{3}{7}, \frac{\sqrt{58}}{7}; \frac{\sqrt{58}}{7}, \frac{3}{\sqrt{58}}, \frac{\sqrt{58}}{7}$, sec A.

EXAMPLES V. (page 20).

1. ·24. 2. ·31; ·95. 3. 1·38; ·73.
4. 1·06. 5. 2·46. 6. ·92; ·38.
7. ·97; ·23; 4·33. 8. 1·26.
9. ·87. 10. 1·20. 11. 28°.
12. 45°. 13. 70°.
14. As the angle increases, the sine increases.
15. 18°. 16. 37°. 17. 41°.
18. 34°. 19. 56°. 20. 35°.

21. $\frac{2}{\sqrt{29}}$, $\frac{5}{\sqrt{29}}$. 22. $\frac{\sqrt{91}}{3}$, $\frac{10}{3}$.

23. $\frac{3}{5}$, $\frac{4}{3}$. 24. $\frac{3}{\sqrt{55}}$, $\frac{\sqrt{55}}{8}$.

25. $\frac{b}{\sqrt{a^2+b^2}}$, $\frac{a}{\sqrt{a^2+b^2}}$. 26. $\frac{4}{3}$. 27. $\frac{25}{24}$.

28. $\cos A = \sqrt{1-s^2}$, $\tan A = \frac{s}{\sqrt{1-s^2}}$, $\operatorname{cosec} A = \frac{1}{s}$,

$\sec A = \frac{1}{\sqrt{1-s^2}}$, $\cot A = \frac{\sqrt{1-s^2}}{s}$.

29. $\sin A = \frac{t}{\sqrt{1+t^2}}$, $\cos A = \frac{1}{\sqrt{1+t^2}}$, $\operatorname{cosec} A = \frac{\sqrt{1+t^2}}{t}$,

$\sec A = \sqrt{1+t^2}$, $\cot A = \frac{1}{t}$.

30. $-\frac{135}{258}$. 31. $\frac{1+a^2}{\sqrt{2+2a^4}}$, $\frac{\sqrt{2+2a^4}}{1-a^2}$.

32. $\sqrt{\frac{m}{n}}$, $\sqrt{\frac{n}{m+n}}$. 33. $\frac{9}{40}$.

34. $\frac{5}{6}$. 35. $\frac{x^4-x^2y^2+y^4}{y^2(y^2-x^2)}$. 36. 1.

37. 0. 38. $\frac{a\sqrt{q^2-p^2}-p\sqrt{q^2-a^2}}{q^2}$.

39. 0. 40. $\frac{1}{\sqrt{3}}$.

Examples VI. (page 27).

51. $x^2+y^2=r^2$.

52. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

53. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

54. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

55. $x^2+y^2=x'^2+y'^2$.

Examples VII. (page 38).

15. 2.
16. $\dfrac{\sqrt{3}}{4}$.
17. $2\frac{7}{12}$.
18. $1\frac{3}{4}$.
19. $\dfrac{3\sqrt{2}}{2}$.
20. 9.

Examples VIII. (page 39).

1. ·3057; 3·2705.
2. ·4040; 2·4751.
3. ·5736; 1·4281.

Examples IX. (page 42).

1. 30°.
2. 45°.
3. 60°.
4. 60°.
5. 60°.
6. 30°.
7. 30°.
8. 60°.
9. 30° or 16° 6′.
10. 45° or 63° 26′.
11. 45°.
12. 30°.
13. 60°.
14. 30°.
15. 30°.
16. 41° 49′; 19° 28′.
17. 0.
18. 30°.
19. 19° 28′.
20. 30°.
21. 30°.
22. 30° or 60°.
23. 30° or 60°.
24. 58° 18′.
25. 19° 28′ or 90°.

EXAMPLES X. (page 52).

1. 30°.
2. 25 yds.
3. $40\sqrt{3}$, $20\sqrt{3}$ or 69·28, 34·64 ft.
4. $25\sqrt{2}$ or 35·35 yds.
5. 1 mile.
6. 346 ft.
7. $\frac{40\sqrt{3}}{3}$ or 23·09 ft.
8. $100\sqrt{3}$ or 173·2 ft.
9. 137·38 ft.
10. 83·91 yds.
11. 137·2 ft.
12. 63·4 yds.; 36·6 yds., 63·4 yds.
13. 21·162 ft.
14. $\frac{1}{2}$ a mile.
15. 4·732 miles.
16. 23·09; 23·09 ft.
17. 321.
18. 72; 577 ft.
19. 237 ft.
20. 74 ft.
21. 758 metres.
22. 3·6 miles.
23. 398 decimetres.
24. 153 ft.
25. 59° 45′.
26. 83 ft.
27. 41 ft.
28. 79 ft.
29. 1·0029 miles.
30. 119 ft.

EXAMPLES XI. (page 56).

1. 6·7 kilometres.
2. 7·1508; 8·5686; 9·3342; 10·4604.
3. 10 miles; E. 19° 52′ S.
4. 9·239 miles.
5. 668·2 yds.
6. E. 13° S.
7. 50 miles; E. 30° 38′ S.
8. N. 67° 23′ E.; 29·03 miles.
9. 1677 yds.; 434 yds.
10. 2·022 kilometres.
11. 9·8 miles.
12. 1·5557 miles.

EXAMPLES XII. (page 64).

1. 2nd, 3rd.
2. 3rd, 2nd.
3. 4th, 1st.
4. 2nd, 4th.
5. 3rd, 1st.
6. 1st, 3rd.
7. 1st, 3rd.
8. 4th, 2nd.
9. 3rd, 1st.

	sine	tangent	sine	tangent
10.	+	−	−	+
11.	−	+	+	+
12.	+	−	+	−
13.	−	+	−	−
14.	−	−	−	+
15.	+	−	−	+
16.	−	−	−	−
17.	−	+	−	−
18.	+	−	−	+
	cosine	**cosecant**	**cosine**	**cosecant**
19.	−	−	−	+
20.	−	−	−	+
21.	+	−	−	−
22.	−	+	−	+
23.	−	+	−	−
24.	−	+	+	−

EXAMPLES XIII. (page 82).

1. $\frac{1}{\sqrt{2}}$.
2. $-\frac{1}{\sqrt{2}}$.
3. $-\sqrt{3}$.
4. $\sqrt{3}$.
5. $-\sqrt{2}$.
6. 2.
7. $-\frac{1}{\sqrt{2}}$.
8. $\frac{\sqrt{3}}{2}$.
9. $-\frac{1}{2}$.
10. $-\frac{1}{\sqrt{2}}$.
11. $\sqrt{3}$.
12. 1.
13. $\frac{2}{\sqrt{3}}$.
14. $\frac{1}{\sqrt{3}}$.
15. 2.
16. $\frac{1}{\sqrt{2}}$.
17. $-\frac{2}{\sqrt{3}}$.
18. $-\frac{1}{2}$.

19. $\frac{2}{\sqrt{3}}$. 20. $-\frac{2}{\sqrt{3}}$. 21. $-\frac{2}{\sqrt{3}}$.

22. $-\frac{1}{\sqrt{3}}$. 23. $\sqrt{3}$. 24. $-\frac{1}{2}$.

25. 1. 26. -1. 27. $\frac{\sqrt{3}}{2}$.

28. $\frac{1}{2}$. 29. 1. 30. $\sqrt{3}$.

31. $\frac{1}{\sqrt{2}}$. 32. $\sqrt{2}$. 33. $-\frac{1}{2}$.

34. $\frac{1}{\sqrt{2}}$. 35. $\sqrt{3}$. 36. $-\frac{2}{\sqrt{3}}$.

37. 1. 38. 30°, 330°. 39. 30°, 150°.

40. 45°, 225°. 41. 150°, 330°. 42. 120°, 240°.

43. 225°, 315°. 44. $\frac{3}{5}$, $-\frac{5}{4}$. 45. $-\frac{3}{5}$, $-\frac{3}{4}$.

46. $\frac{12}{13}$, $\frac{12}{5}$. 47. $\frac{7}{25}$, $\frac{24}{7}$. 48. $-\cot A$.

49. $\sin A$. 50. $\cot A$. 51. $\sec A$.

52. $\sin^2 A$. 53. $\sin A$. 54. $\operatorname{cosec} A$.

55. $\operatorname{cosec} A$. 56. $-\sec A \operatorname{cosec} A$. 57. $-\sin A \sec^2 A$.

58. $-\operatorname{cosec}^2 A$. 67. 0. 68. $\frac{\sqrt{3}}{2}$.

69. $\frac{\sqrt{3}}{2}$. 70. -1. 71. $-\frac{1}{2}$.

72. 0. 73. $\frac{1}{3}$. 74. 1.

75. 0·46. 76. 1·92. 77. ·81.

78. 1·56 secs.

EXAMPLES XIV. (page 91).

1. 2, 0, 1, 3, 3, 4.
2. $\bar{1}$·8834, 2·8834, $\bar{2}$·8834, $\bar{4}$·8834, 4·8834, ·8834.
3. ·3735, 37·35, ·03735, ·00003735, 373500.
4. 5, 10, 14. 5. 5th, 8th, 11th.
6. ·2353, ·2156, ·4117, 1·2938, 6·5876, 13·2345.
7. ·1990, ·3861, ·7553, 1·8416.
8. 5·1369, 5·0653, 2·8761, 1·8015, 9·1955.

Examples XV. (page 99).

1. 5929.
2. $2{\cdot}801 \times 10^{7}$.
3. $1{\cdot}513 \times 10$[illegible].
4. 3·702.
5. 2·794.
6. 5·982.
7. 48·68.
8. $5{\cdot}940 \times 10^{3}$.
9. 61·28.
10. $2{\cdot}755 \times 10^{6}$.
11. 4·768 secs.
12. $1{\cdot}305 \times 10^{9}$ cu. cm.
13. 2656 cms.
14. 5·27 cms.
15. 127·6.
16. $2{\cdot}471 \times 10^{3}$ cu. dm.
17. 775·6.
18. ·1276 gram.
19. $1{\cdot}413 \times 10^{6}$ cms.
20. ·1757.
21. $-$ ·96.
22. $-$ ·82.
23. 5·78.
24. 15·30.
25. $9{\cdot}594 \times 10^{13}$ miles.
26. $2{\cdot}558 \times 10^{13}$ miles.
27. 4·347 years.
28. 2·6025, 2·6039.
29. 2·5201, 2·5221.

Examples XVI. (page 105).

1. ·1852.
2. ·5088.
3. 2·026.
4. 4·389.
5. ·7144.
6. 1·489.
7. 1·357.
8. $-$3·959.
9. $-$1·094.
10. ·8910.
11. 1·689.
12. ·8535.
13. 210·4 sec.
14. ·2882.
15. 82° 20′.
16. ·001139.
17. ·1888 hours.
18. 44·71 feet.
19. 44·74 feet.
20. 309·0 feet per second.

Examples XVII. (page 119).

1. 106·7 sq. cm.
2. 219·5 sq. cm.
3. 123·4 sq. cm.
4. 11·49.
5. 17·71.
6. 14·13.
7. ·9645.
8. ·5631.

Examples XVIII. (page 122).

1. $A = 37° 5'$, $a = 76{\cdot}70$, $b = 101{\cdot}5$.
2. $B = 52° 38'$, $a = 95{\cdot}43$, $c = 157{\cdot}3$.
3. $A = 31° 8'$, $B = 58° 52'$, $b = 27{\cdot}65$.

4. $A = 49° 19'$, $B = 40° 41'$, $c = 41·28$.
5. $A = 58° 51'$, $B = 31° 9'$ ($= 31° 8'·5$), $a = 202·2$.
6. $B = 56° 38'$, $a = 16·44$, $b = 24·98$.
7. $B = 74° 43'$, $a = 7·466$, $c = 28·32$.
8. $A = 16° 46'$, $B = 73° 14'$, $b = 788·3$.
9. $A = 68° 55'$, $B = 21° 5'$, $c = 344·4$.
10. $A = 7° 47'$, $b = 12·67$, $c = 12·79$.
11. $A = 30° 46'$, $a = ·7838$, $b = 1·316$.
12. $A = 41° 15'$, $B = 48° 45'$, $c = 23·03$.

Examples XIX. (page 134).

1. $A = 143° 49'$.
2. $B = 102° 39'$.
3. $A = 104° 15'$.
4. $A = 106° 37'$.
5. $C = 102° 33'$.
6. $C = 97° 9'$.
7. $B = 12° 39'$.
8. $C = 39° 42'$.
9. $B = 35° 36'$.
10. $B = 36° 22'$ or $36° 21'$.
11. $A = 65° 1'$; $B = 52° 20'$; $C = 62° 39'$.
12. $A = 70° 22'$; $B = 55° 39'$; $C = 53° 59'$.

Examples XX. (page 135).

1. $B = 118° 37'$, $C = 31° 45'$, $a = 20·95$.
2. $A = 64° 21'$, $B = 77° 25'$, $c = 27·39$.
3. $B = 30° 28'$, $C = 90° 55'$, $a = 46·02$.
4. $A = 66° 39'$, $C = 87° 8'$, $b = 14·34$.
5. $A = 64° 19'$, $B = 78° 16'$, $c = 10·6$.
6. $B = 76° 18'$, $C = 41° 26'$, $a = 48·21$.
7. (i) $A = 36° 10'$, $B = 91° 37'$; (ii) $A = 36° 9'$, $B = 91° 37'$.
8. (i) $B = 57°$, $C = 49° 48'$; (ii) $B = 57°$, $C = 49° 48'$.
9. (i) $A = 28° 46'$, $C = 115° 32'$; (ii) $A = 28° 45'$; $C = 115° 32'$.
10. (i) $B = 69° 59'$, $C = 92° 39'$; (ii) $B = 70° 1'$, $C = 92° 37'$.
11. (i) $A = 87° 18'$, $B = 39° 28'$; (ii) $A = 87° 17'$, $B = 39° 29'$.
12. (i) $A = 93° 36'$, $C = 41° 57'$; (ii) $A = 93° 36'$, $C = 41° 56'$.

Examples XXI. (page 136).

1. Not ambiguous. 2. Ambiguous. 3. Ambiguous.
4. B $=74°\,15'$ or $105°\,45'$, C $=50°\,31'$ or $19°\,1'$.
5. A $=79°\,42'$ or $100°\,18'$, B $=37°\,27'$ or $16°\,51'$.
6. B $=58°\,37'$, C $=49°\,8'$.
7. A $=82°\,6'$ or $6°\,50'$, C $=52°\,22'$ or $127°\,38'$.
8. B $=37°\,27'$ or $142°\,33'$, C $=106°\,50'$ or $1°\,44'$.
9. A $=38°\,19'$, B $=82°\,4'$.
10. A $=96°\,10'$ or $9°\,22'$, C $=46°\,36'$ or $133°\,24'$.
11. B $=25°\,30'$, C $=82°\,15'$, $c=190$.
12. A $=87°\,56'$ or $7°\,10'$, C $=49°\,37'$ or $130°\,23'$, $a=108$ or $13{\cdot}48$.

Examples XXII. (page 137).

1. $a=35{\cdot}32$, $b=107{\cdot}3$. 2. $a=24{\cdot}65$, $c=30{\cdot}30$.
3. $b=31{\cdot}25$, $c=41{\cdot}90$. 4. $b=14{\cdot}51$, $c=14{\cdot}69$.
5. $a=28{\cdot}12$, $c=22{\cdot}35$. 6. $a=43{\cdot}01$, $b=37{\cdot}38$.
7. $a=59{\cdot}64$, $c=49{\cdot}49$. 8. $b=25{\cdot}07$, $c=26{\cdot}55$.
9. $b=116{\cdot}0$, $c=148{\cdot}4$. 10. $a=783{\cdot}9$, $b=788{\cdot}9$.

Examples XXIII. (page 137).

1. $50°\,33'$. 2. $59°\,52'$, $66°\,41'$.
3. Ambiguous, Ambiguous, Non-ambiguous.
4. B $=74°\,45'$ or $105°\,15'$; A $=56°\,21'$ or $25°\,51'$; $a=15{\cdot}9$ or $8{\cdot}326$. 5. $b=9{\cdot}603$, $c=17{\cdot}18$. 6. $66°\,59'$, $40°\,24'$.
7. $159°$. 8. B $=51°\,29'$ or $128°\,31'$; C $=84°\,14'$ or $7°\,12'$.
9. $b=79{\cdot}21$, $c=84{\cdot}22$.
10. B $=39°\,14'$ or $140°\,46'$, C $=105°\,32'$ or $4°$, $c=32{\cdot}48$ or $2{\cdot}352$.
11. $67°\,35'$. 12. $18°\,36'$. 13. B $=27°\,6'$, C $=89°\,39'$.
14. $a=37{\cdot}08$, $b=46{\cdot}35$. 15. $87°\,47'$, $43°\,41'$, $21{\cdot}37$.
16. $109°\,39'$. 17. $b=325{\cdot}7$, $c=248{\cdot}5$.
18. A $=49°\,45'$, C $=58°$. 19. $41°$, $54°\,28'$, $84°\,32'$.
20. $74°\,52'$, $51°\,42'$. 21. $b=21{\cdot}47$, $c=5{\cdot}802$. 22. $31°\,48'$.
23. A $=93°\,55'$ or $5°\,9'$; C $=45°\,37'$ or $134°\,23'$.
24. $a=17{\cdot}55$, $b=15{\cdot}11$. 25. $141{\cdot}5$. 26. $36°\,30'$.
27. $105{\cdot}8$, $74{\cdot}94$. 28. $97{\cdot}27$. 29. $45{\cdot}54$. 30. $99{\cdot}68$.

Examples XXIV. (page 142).

1. 48·05 metres. 2. 7·193 miles. 3. 3337 ft.
4. 124·2 ft. 5. 1835 ft. 6. Height = 969 ft.; distance = 1863 ft. 7. 267·7 yds. 8. 4381 ft.
9. 5·487 metres. 10. 127·4 ft. 11. 753·4 yds.
12. 185·7 yds. 13. 971·2 ft. 14. 563·6 yds.
15. 613 ft. or 1558 ft. 16. 23·58 ft.
17. 869 ft.

Examples XXV. (page 145).

1. 101·4 ft. 2. 8535 yds. 3. 1830 yards.
4. 1114 ft. 5. 5293 ft. 6. 118·1 ft. 7. 122 ft.
8. 114 ft. 9. 3768 ft. 10. 195·1 ft.

Examples XXVI. (page 151).

1. 159·4 ft. 3. 3680 ft. 4. 1614 yds.
5. 111·9 yds. 6. E. 37° 48′ N. 7. $\frac{abc \sin a}{2A}$.
8. 801·98 ft. 9. 9·451 miles. 10. 20·65 mls. per hour.
11. 1927 ft. 12. 5537 ft. 13. 560·6 ft.
14. 157 ft.; 546 ft. 15. 4067 mls.
16. Tower = 190·6 ft., Flagstaff = 87·34 ft.
17. 81·76 ft. 19. 193·9 ft.
20. 1195 ft. 21. 243·1 ft. 22. 296·2 ft.
24. 3293 ft. 25. 57·40 ft. 26. 121·2 yds.
27. 91·9 ft. 28. 2502 ft. 29. 354·4 yards.

Examples XXVII. (page 159).

1. 2403 miles. 2. 14°·03. 3. 17·41 miles.
4. 32·40 miles. 5. 4·735 mls. per hour. 6. 238·1 ft.

Examples XXVIIa. (page 159a).

1. 2° 52′. 2. 3980$\frac{2}{11}$ miles.
3. 2236·8 miles; 3911$\frac{1}{2}$ miles. 4. 3969·4 miles.
5. 1° 9′. 6. 27″ 3‴.
7. (i) 2082 miles, (ii) 1161 miles, (iii) 1107 miles.
8. 39° 58′, 36° 52′. 9. 169·2 hrs. 10. 960 miles; 8 min 16″ miles.

Examples XXVIIb. (page 159a).

1. 72° 39′, 53° 24′, 41° 50′. 2. 1·515 ft.; 11° 6′.
3. 18° 59′, 14° 7′. 4. 29° 28′. 5. 35° 16′; 60°.
6. 50° 28′, 72° 27′, 64° 46′, 71° 34′, 95° 44′.
7. 28° 22′, 41° 8′. 8. 5·58 yards. 9. 3·687 in.; 21° 38′.
10. 75·75 yd. 11. 68° 39′. 12. 21 in 100.

Examples XXVIII. (page 167).

27. $\frac{\sqrt{3}+1}{2\sqrt{2}}$; $\frac{\sqrt{3}+1}{2\sqrt{2}}$. 28. $\frac{\sqrt{3}+1}{2\sqrt{2}}$; $-\frac{\sqrt{3}+1}{2\sqrt{2}}$.

29. $\frac{\sqrt{3}-1}{2\sqrt{2}}$; $\frac{\sqrt{3}-1}{2\sqrt{2}}$.

Examples XXIX. (page 171).

13. $\cos A \cos B \cos C\,(\tan A + \tan B + \tan C - \tan A \tan B \tan C)$.
14. $\cos A \cos B \cos C\,(1 - \tan B \tan C - \tan C \tan A - \tan A \tan B)$.
15. $\frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$.
16. $\cos A \cos B \cos C\,(\tan A + \tan B - \tan C + \tan A \tan B \tan C)$.
17. $\cos A \cos B \cos C\,(1 + \tan B \tan C - \tan C \tan A + \tan A \tan B)$.
18. $\frac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C + \tan C \tan A + \tan A \tan B}$.

Examples XXX. (page 178).

4. (i) $\frac{\sqrt{3}}{2}$; $\frac{1}{2}$; $\sqrt{3}$. (ii) $\frac{4\sqrt{2}}{9}$; $\frac{7}{9}$; $\frac{4\sqrt{2}}{7}$.
(iii) $\frac{\sqrt{15}}{8}$; $\frac{7}{8}$; $\frac{\sqrt{15}}{7}$. (iv) $\frac{24}{25}$; $\frac{7}{25}$; $\frac{24}{7}$.
(v) $\frac{120}{169}$; $-\frac{119}{169}$; $-\frac{120}{119}$.

5. (i) $\frac{\sqrt{3}}{2}$; $\frac{1}{2}$; $\sqrt{3}$. (ii) 1; 0; ∞.
(iii) $\frac{24}{25}$; $-\frac{7}{25}$; $-\frac{24}{7}$. (iv) $\frac{120}{169}$; $\frac{119}{169}$; $\frac{120}{119}$.
(v) $\frac{\sqrt{35}}{18}$; $-\frac{17}{18}$; $-\frac{\sqrt{35}}{17}$.

6. (i) $\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$; $-\sqrt{3}$. (ii) 1; 0; ∞.
(iii) $\frac{24}{25}$; $\frac{7}{25}$; $\frac{24}{7}$. (iv) $\frac{120}{169}$; $\frac{119}{169}$; $\frac{120}{119}$.
(v) $\frac{28}{53}$; $\frac{45}{53}$; $\frac{28}{45}$. (vi) $\frac{33}{65}$; $\frac{56}{65}$; $\frac{33}{56}$.

7. (i) $\frac{1}{2}$. (ii) $\frac{1}{4}$. (iii) $\frac{1}{8}$. (iv) $\frac{1}{8}\sqrt{2}$.

EXAMPLES XXXI. (page 185).

1. $2 \sin \frac{3A}{2} \cos \frac{A}{2}$.
2. $2 \cos \frac{5A}{2} \cos \frac{A}{2}$.
3. $2 \sin 6A \sin A$.
4. $2 \cos 4A \sin A$.
5. $2 \sin 8A \cos 3A$.
6. $2 \cos 4A \cos A$.
7. $2 \sin 2A \sin 3A$.
8. $-2 \cos 5A \sin 2A$.
9. $2 \sin 46° \cos 16°$.
10. $2 \sin 45° \sin 10°$.
11. $2 \cos 39° \cos 3°$.
12. $2 \cos 42° \sin 10°$.
13. $2 \cos 37° \cos 14°$.
14. $2 \sin 13° \cos 2°$.
15. $-2 \cos 36° \sin 13°$.
16. $-2 \sin 47° \sin 5°$.

EXAMPLES XXXII. (page 186).

1. $\sin 3A + \sin A$.
2. $\cos 3A + \cos A$.
3. $\sin 5A - \sin 3A$.
4. $\cos 2A - \cos 4A$.
5. $\sin 12A - \sin 4A$.
6. $\cos 12A + \cos 2A$.
7. $\sin 8A - \sin 2A$.
8. $\cos 2A - \cos 8A$.
9. $1 - \sin 30° = \cdot 5$.
10. $\cos 98° + \cos 8° = \cdot 8511$.
11. $\frac{1}{2}[\sin 80° - \sin 10°] = \cdot 4056$.
12. $\cos 120° + \cos 20° = \cdot 4397$.
13. $\frac{1}{2}[\cos 23° - \cos 127°] = \cdot 7612$.
14. $\sin 95° + \sin 15° = 1 \cdot 2550$.
15. $\frac{1}{2} \cos 26° = \cdot 4494$.
16. $\frac{1}{2}[\sin 213° - \sin 67°] = -\cdot 7326$.
17. $\cos 3A + \cos (A + 2B)$.
18. $-\cos (3A + 8B) + \cos (A + 2B)$.
19. $\sin (4x + 6y) + \sin (2x + 2y)$.
20. $\sin (4x + 4y) - \sin (2x + 6y)$.

EXAMPLES XXXIV. (page 199).

12. $73° 2'$; $12 \cdot 56$.
13. $33° 34'$; 30
14. $58° 44'$; $38 \cdot 53$.

TEST PAPERS.

I. (page 201).

1. ·1917962.
2. ·2619; $137\frac{5}{11}°$.
3. 78·5714 sq. m.
4. $\frac{3}{5}$.
5. $1\frac{149}{315}$ cms.
6. ·79; ·62.
7. 46°.
8. ·94; ·35.

II. (page 202).

1. 65° 12′ 18″.
2. 150°; ·5587.
3. 44 cms.; 154 sq. cms.
4. $95\frac{5}{11}°$.
5. 12 sq. cms.
6. 70°.
7. ·48; ·87; ·55.
8. 1·091; ·436.

III. (page 202).

1. ·5808950617283.
2. $91\frac{7}{11}°$; 1·3095.
3. $3\frac{3}{11}$ metres.
4. 3·99 metres.
5. $\frac{14}{25}$ radians.
6. ·96; 1·05.
7. 68°.
8. $\frac{4}{5}$; $\frac{4}{3}$; $\frac{5}{4}$.

IV. (page 203).

1. 112° 3′.
2. ·832; 1·803.
3. 36°.
4. 60 mm.; 113 sq. cms.
5. 4·4 cms.
6. $43\frac{1}{22}°$.
7. $35\frac{35}{44}°$.
8. $\frac{10}{17}$.

V. (page 203).

1. $\cos A = \frac{35}{37}$; $\sec A = \frac{37}{35}$;
   $\tan A = \frac{12}{35}$; $\cot A = \frac{35}{12}$;
   $\operatorname{cosec} A = \frac{37}{12}$.
2. 1·309 metres.
3. $-$·25.
4. ·32.
5. ·57; ·82.
6. ·476.
7. 50·29 sq. cms.

VI. (page 204).

1. $74° 12' 18''$. 2. $82°$.
3. $128\frac{4}{7}°$; 2·24. 4. 2·95.
5. $\sin A = \cdot 39$; $\operatorname{cosec} A = 2\cdot 54$;
$\cos A = \cdot 92$; $\sec A = 1\cdot 09$;
$\cot A = 2\cdot 33$.
6. ·51; 1·12. 7. 1·39.
8. $98° 44' 17''$.

VII. (page 205).

1. $32\frac{8}{11}°$; ·57. 2. 1·19; 1·56.
3. $29°$. 4. $65°$.
5. $\sin 32° = \cdot 53$; $\sin 58° = \cdot 85$;
$\cos 32° = \cdot 85$; $\cos 58° = \cdot 53$.
6. $98\frac{2}{11}°$, $49\frac{1}{11}°$, $32\frac{8}{11}°$.
7. $60°$; 1·05. 8. 5·4 cms.

VIII. (page 205).

1. $132°$; 2·30. 2. ·78; ·63.
3. $11°$. 4. 10309·09 sq. cms.
5. ·8; ·6. 6. $\sin 24° = \cdot 41$;
$\cos 24° = \cdot 91$;
$\sin 48° = \cdot 74$.
7. $162\frac{6}{7}°$; 2·84. 8. 186 sq. m.

IX. (page 206).

1. $\frac{135}{126}$; $\frac{116}{126}$. 2. ·54; ·84; ·84; ·54.
4. ·89.
5. 342, 500, 642·8, 766, 866, 939·7, 984·8, 1000.
6. 1·83. 8. $90°$.

X. (page 207).

1. $1071\frac{4}{7}$ mls. 2. $6000°$.
3. $38° 11'$; ·7865; ·62. 5. $55° 9'$; ·8207.
6. $4\frac{1}{2}$. 7. $30°$.
8. 1·9443, 2·8986, 3·8304, 4·7331, 5·6.

XI. (page 208).

1. ·86594688 ; ·8660.
2. 155° 36′ 36″.
3. Each = ·53 ; B = 32°.
4. −1, −·60, −·12, ·37, ·81.
6. 60°.
8. $\sin 70° < 2 \sin 35°$.

XII. (page 208).

2. 60°, 180°.
3. ·02619 ; 1·0004.
6. 69·5 miles.
7. $\frac{\sqrt{3}}{2} = \cdot 8660$.
8. ·0718.

XIII. (page 209).

2. 45°, 33° 42′.
3. 864000 miles.
4. 2·973.
5. 17°; 3·27.
6. 2, 2·14, 2·22, 2·23, 2·17, 2·05.
7. 7159 sq. metres.
8. ·57, ·30, ·95.

XIV. (page 210).

2. 1·08, 1·20.
3. $\frac{84}{13}$.
4. 36·082.
5. 16° 21′ $49\frac{1}{11}$″.
6. 45°, 60°; 30°, 90°.
7. 564·1 metres.

XV. (page 211).

2. 30°, 45°; 60°.
3. $(1-p)/(1+p)$.
4. $\frac{3}{5}$.
5. $\frac{121m}{1260}$.
6. $\frac{1}{12}$.
7. ·766022 ; cos 40° = ·7660.

XVI. (page 212).

2. 60°; 60°.
3. 3.
4. 3·1416.
5. Large hand 510° or 8·90 radians;
   Small hand $42\frac{1}{2}$° or ·74 radian.
6. ·6427802 ; cos 50° = ·6428.
7. $\frac{1}{4}$.

XVII. (page 212).

1. 1·57, 1·89, 2·10.
3. tan 10° = OP/OA = ·18;
   tan 20° = OQ/OA = ·36;
   tan 40° = OR/OA = ·84.
4. 5·9524 miles.
5. cos 25° = ·9063; cos 45° = ·7071.
6. 60°, $\frac{\pi}{3}$; 132° 3′, $\frac{\pi}{3}$.
7. ·744.
8. ·629.

XVIII. (page 213).

2. (i) 30°, 150°; (ii) 30°, 135°, 150°, 315°.
3. 65·61 metres.
6. 10′ 40″ after 1.
7. $\frac{240}{289}$, $\frac{161}{289}$, $\frac{240}{161}$.
8. 8·944, 4·472, 17·889 feet.

XIX. (page 214).

2. (i) 30°, 210°; 120°, 300°; (ii) 60°, 240°; 150°, 330°.
3. 91 metres.
5. $\frac{63}{65}$.
6. 2723$\frac{17}{21}$ miles; 3108·4 miles.
7. 3·4795.

XX. (page 215).

1. 1976·3 metres.
3. (i) 60°, 240°; 30°, 210°.
   (ii) 30°, 330°; 150°, 210°.
4. 25·98 metres.
5. (i) cos A = $\frac{21}{29}$, cot A = $\frac{21}{20}$;
   (ii) cos A = $-\frac{21}{29}$, cot A = $-\frac{21}{20}$.
7. 336·9; − 222·39.
8. 7 feet.

XXI. (page 216).

1. 88$\frac{44}{49}$ tons.
3. − ·999.
4. 12·7 dms.
5. 21·872 centimetres.
6. (i) 135°, 315°; 90°.
   (ii) 30°, 150°; 210°, 330°.
7. 6087 feet.
8. ·87 ampères.

XXII. (page 217).

2. 45°, 225°; 67° 30′, 247° 30′;
157° 30′, 337° 30′.

3. $\frac{1343}{584}$.

4. 46·13 dms., 99·97 dms.

5. ·7072 miles.

6. 33° or 42°.

7. 589·9 mls. per hr.

8. 20,000 sq. m.

XXIII. (page 218).

1. 1·722, ·01977.

2. 95·66, 111·69 yards.

3. Each ratio = tan 41°
= ·8693.

4. 1·030.

5. 1·05.

6. tan θ = ·6928,
cos θ = ·8219.

7. ·9962, ·9848, ·9659, ·9397, ·9063.

8. 60°, 300°.

XXIV. (page 219).

1. 1210·96 metres.

2. 3·8.

4. 11° 59′; 471·18 ft.

5. ·06469.

6. tan θ = ·3106,
sin θ = ·2965.

7. 24°.

XXV. (page 220).

1. (i) 10·45, (ii) 11·37.

3. ·299.

4. 2·617 grams-weight.

5. 3° 20′, 7° 5′.

6. 3·484, 45·03 cms.

7. (i) 103° 33′, 256° 27′,
(ii) 24° 38′, 155° 22′.

8. tan θ = ·6249, 2·1693;
θ = ·5587, 1·1393;
sin θ = ·5299, ·9082.

XXVI. (page 221).

1. 28·94 cms.
3. 7·914, ·006516.
4. sin 156° = ·4067, cos 156° = − ·9135;
sin 204° = − ·4067, cos 204° = − ·9135;
sin 336° = − ·4067, cos 336° = ·9135;
sin 114° = ·9135, cos 114° = − ·4067;
tan 114° = − 2·246.
5. 73·62 dms.
6. (i) 36° 52′, 143° 8′;
(ii) 45°, 225°; 18° 26′, 198° 26′.
7. $1{\cdot}015 \times 10^{13}$.
8. 22° 30′.

XXVII. (page 222).

1. 2·985.
2. 1·259.
4. Height = 54·39 ft.
Distance from one post = 85·37 ft.
6. $\sin\theta = {\cdot}6691, \quad \theta - \frac{\theta^3}{4} = {\cdot}6347;$

$\cos\theta = {\cdot}7431, \quad 1 - \frac{\theta^2}{2} = {\cdot}7311.$
7. 100° 13′ 38″.
8. 30°, 210°, 150°, 330°.

XXVIII. (page 223).

2. 714·8.
3. ·3121.
4. 2·52 dms.
5. 11° 19′.
6. $\pm\frac{99}{20}$.
7. 51° 8′; 3571 miles.

XXIX. (page 223).

3. B = 89° 5′, C = 53° 7′.
4. (i) 807·2, (ii) $1{\cdot}418 \times 10^{5}$.
5. 70° 20′.
6. 90°.
7. 45°, 75°, 105°, 135°;
·785, 1·309, 1·833, 2·356.

XXX. (page 224).

2. 121·9 ft.
3. $90°, 270°; 60°, 120°; 240°, 300°$.
5. 1·443.
6. $50° 10'$.
7. $67° 52'$ or $112° 8'$.

XXXI. (page 225).

1. $0°; 60°, 300°$.
3. 85·76 cms.
4. (i) and (iii) ambiguous.
5. (i) 417·2, (ii) 23·44.
6. $A = 27° 45'$, $C = 110° 54'$, $b = 58·70$.
7. $\frac{9}{2}$.

XXXII. (page 225).

2. 320 ft.; 135·3 ft.
3. $AB = 17·69$, $AC = 23·73$; $BC = 27·77$ cms.
4. 3·118 cms.
5. 18·49 mls. per sec.
6. $A = 101° 47'$, $B = 24° 31'$, $a = 29·04$ cms.

XXXIII. (page 226).

2. 12·4 sq. cms.
3. 11·02 kilometres.
4. 2·379.
5. $44° 17'$ or $135° 43'$.
6. 215 cms. (using arc $= r\theta$).
7. $0°, 240°$.

XXXIV. (page 227).

1. $x = 5$, $y = 2$.
2. $-1·9468$.
3. $48° 11'$, $73° 24'$.
4. 30·02 metres.
5. 1·126 metres.
7. $21° 20'$.

XXXV. (page 228).

1. 509·9 metres.
2. 42·23, 13·53 cms.
3. 1·97 sq. cms.
4. 20·78.
7. $2·21 \times 10^7$ kilograms.

XXXVI. (page 228).

1. 24·54 metres.
2. 73° 41′.
3. 90·23.
4. 1163 metres.
6. A = 87° 3′, B = 63° 41′.
7. 16° (17° using logs).

XXXVII. (page 229).

1. Volume = 316·31 cc.
2. (i) 11·32, (ii) 4·536.
3. 64·28 cms.; ·06 cms.
4. 15·39 metres.
5. $(a-1)(a-2)$.
6. 1·280 cms.

XXXVIII. (page 230).

1. 3·61; 60°.
2. 2562 feet.
3. 30°, 150°, 210°, 330°.
6. 1·741.
7. 46° 31′ or 133° 29′.

XXXIX. (page 231).

1. (i) 1·513.
   (ii) 71° 26′.
3. 361·4 metres.
4. 1° 16′.
6. 4·492 mls. per hr.,
   1·666 miles.
7. 1·615 × 10⁴.

XL. (page 231).

2. [illegible] or 47 radians approx.
5. [illegible]
6. [illegible]

XLI. (page 232).

1. A = 201° 30′; B = 338° 30′, C = 316° 10′.
3. 576·9, 516·8.
6. 3·461.
7. 1; $\frac{1}{n}$.

XLII. (page 233).

5. (i) $52^\circ\, 2'$; $127^\circ\, 58'$,
   (ii) $134^\circ\, 45'$.
7. $(-1)^{n+1}\, 4 \sin n\text{A} \sin n\text{B} \sin n\text{C}$.

XLIII. (page 234).

3. (i) $\cos\alpha$ or $\sin\alpha$; (ii) $\cot\beta$ or $-\tan\beta$.
4. 625 metres.

XLIV. (page 235).

3. 51·77 metres; $11^\circ\, 19'$.

XLV. (page 236).

4. $\cos\theta = \dfrac{1 \pm \sqrt{5}}{4} \cos\alpha$.
6. $\text{A} = 6^\circ$; $\text{B} = 54^\circ$; $\text{C} = 120^\circ$.

ANSWERS.

PART II.

Examples XXXV. (page 244).

30. 100·65 feet.
31. 2 : 1, 1 : 3, 3 : 2.
32. 4·773 centimetres.

Examples XXXVII. (page 262).

1. 19·59 sq. centimetres.
2. 9·6 cms.; 7·4 cms.; 5·0 cms.

Examples XXXVIII. (page 266).

1. 8·6605 cms.; 259·82 sq. cms.
2. 13·254 ft.
3. 5 cms.
4. 105·804 sq. inches.
7. 181 sq. cms.
8. 173 sq. ins.
9. 9·5 cms.
12. 114588 sq. ft.

Examples XXXIX. (page 273).

N.B. In some of these answers more than sufficient is given, *e.g.* in 11, $\frac{2n\pi}{5}$ is sufficient as it embraces $2n\pi$.

1. $\frac{n\pi}{2} + (-1)^n \frac{\pi}{6}$.
2. $\frac{2n\pi}{3} \pm \frac{\pi}{9}$.

3. $\frac{n\pi}{4}+\frac{\pi}{16}$.

4. $360n^\circ+(-1)^n 4^\circ 6'$.

5. $60n^\circ \pm 3^\circ 2'$.

6. $\frac{180n^\circ}{7}+5^\circ 1'$.

7, 8, 9. $\frac{n\pi}{3}\pm\frac{\pi}{9}$.

10. $2n\pi$; $\frac{2n+1}{3}\pi$.

11. $(2n\pi)$; $\frac{2n\pi}{5}$.

12. $n\pi$.

13. $\frac{n\pi}{6}$; $\frac{n\pi}{4}$.

14. $(n\pi)$; $\frac{n\pi}{7}$.

15. $\left(\frac{n\pi}{2}\right)$; $\frac{n\pi}{4}$.

16. $2n\pi-\frac{\pi}{2}$; $\frac{2n\pi}{5}+\frac{\pi}{10}$.

17. $\frac{n\pi}{4}+\frac{\pi}{16}$; $n\pi+\frac{\pi}{4}$.

18. $\frac{n\pi}{9}+\frac{\pi}{18}$.

19. $\frac{n\pi}{3}$; $2n\pi\pm\frac{\pi}{3}$.

20. $\frac{n\pi}{4}$; $\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$.

21. $\frac{n\pi}{2}\pm\frac{\pi}{8}$; $n\pi+(-1)^n 14^\circ 29'$.

22. $\frac{n\pi}{2}\pm\frac{\pi}{8}$; $n\pi\pm 22^\circ 21'$.

23. $\left(2n\pi\pm\frac{\pi}{2}\right)$; $\frac{n\pi}{4}$; $\left(\frac{2n+1}{4}\pi\right)$.

24. $2n\pi\pm\frac{\pi}{2}$; $\frac{n\pi}{6}+\frac{\pi}{24}$; $\frac{n\pi}{2}-\frac{\pi}{8}$.

25. $\frac{n\pi}{8}$; $\left(\frac{n\pi}{4}\right)$.

26. $\left(\frac{n\pi}{4}\right)$; $\frac{n\pi}{12}$.

27. $\frac{n\pi}{2}$; $\frac{(2n+1)\pi}{8}$.

28. $n\pi$; $\frac{2n+1}{10}\pi$.

29. $360n^\circ+63^\circ 50'$; $360n^\circ-20^\circ 14'$.

30. $360n^\circ+74^\circ 44'$; $360n^\circ-33^\circ 38'$.

31. $360n^\circ+36^\circ 52'$.

32. $360n^\circ+31^\circ 54'$; $2n\pi$.

33. $2n\pi+\frac{\pi}{3}$; $(2n+1)\pi$.

34. $2n\pi+\frac{\pi}{4}$.

35. $2n\pi+\frac{5\pi}{12}$; $2n\pi-\frac{\pi}{12}$.

36. $2n\pi+\frac{\pi}{2}$; $(2n+1)\pi+\frac{\pi}{6}$.

37. $n\pi+\frac{\pi}{4}$; $2n\pi$; $2n\pi+\frac{\pi}{2}$.

38. $\frac{n\pi}{2}\pm\frac{\pi}{8}$; $\frac{n\pi}{2}\pm\frac{\pi}{12}$.

39. $2n\pi \pm \frac{\pi}{2}$; $n\pi \pm 41^\circ\, 24'$. 40. $n\pi \pm \frac{\pi}{3}$.

41. $(4n+1)\frac{\pi}{2}$; $(4n \pm 1)\,\pi$. 42. $2n\pi \pm \frac{\pi}{3}$.

43. $n\pi + \frac{\pi}{2}$; $\frac{n\pi}{4} + (-1)^n\, 7^\circ\, 30'$. 44. $n\pi + \frac{\pi}{12}$; $n\pi + \frac{5\pi}{12}$.

45. $\frac{(2n+1)\,\pi}{4} + \frac{3\pi}{16}$. 46. $n\pi \pm \frac{\pi}{6}$; $n\pi \pm \frac{\pi}{4}$; $n\pi$.

47. $2n\pi$; $n\pi + \frac{\pi}{4}$. 48. $n\pi$; $\frac{n\pi}{2} + (-1)^n \frac{\pi}{4}$.

49. $\frac{n\pi - \alpha - \beta}{3}$. 50. $2n\pi \pm \frac{5\pi}{6}$.

51. $2n\pi \pm \frac{2\pi}{3}$. 52. $2n\pi$; $2n\pi \pm \frac{\pi}{2}$; $2n\pi \pm \frac{\pi}{3}$.

53. $n\pi + (-1)^n \frac{\pi}{6}$; $2n\pi + \frac{\pi}{2}$.

54. $n\pi + \frac{\alpha}{2}$; $\frac{(2n+1)\,\pi}{6} - \frac{\alpha}{6}$. 55. $n\pi - \frac{\alpha+\beta}{2}$.

Examples XL. (page 288).

7. $\sin\frac{A}{2} = \frac{3}{5}$, $\cos\frac{A}{2} = -\frac{4}{5}$.

8. $\sin\frac{A}{2} = -\frac{8}{17}$, $\cos\frac{A}{2} = -\frac{15}{17}$.

9. $\sin 130^\circ = 0{\cdot}7660$, $\cos 130^\circ = -0{\cdot}6428$.

10. $\sin 115^\circ = 0{\cdot}9063$, $\cos 115^\circ = -0{\cdot}4226$.

11. $\frac{3}{4}$, $-\frac{4}{3}$.

Examples XLI. (page 296).

1. $\frac{12}{5}$. 2. $\frac{3}{4}$. 3. $\frac{63}{65}$. 4. $\frac{56}{65}$.

EXAMPLES XLII. (page 298).

1. $\frac{1}{3}$ or -1.
2. a or a^2-a+1.
3. $\dfrac{5\pm\sqrt{5}}{2}$.
4. $\dfrac{-1\pm\sqrt{b^2+2}}{a}$.
5. $\sqrt{\dfrac{10\pm4\sqrt{2}}{17}}$.
6. $\dfrac{ab\{\sqrt{a^2-1}-\sqrt{b^2-1}\}}{a^2-b^2}$.
7. 2 or -1.
8. 13.
9. $\dfrac{\sqrt{3}}{2}$.
10. 0, $\frac{1}{2}$.
11. $\dfrac{3\sqrt{3}\pm\sqrt{7}}{8}$.
12. $\frac{156}{5}$.
13. $\frac{1}{4}$ or -8.
14. $\sqrt{3}$, $-(2+\sqrt{3})$.
15. ab.
16. $x=\dfrac{a+b+c\pm\sqrt{a^2+b^2+c^2-ab-bc-ca+3}}{3}$.

EXAMPLES XLIII. (page 303).

1. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
2. $x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=1$.
3. $b^2(h^2+k^2)=(a^2-ch)^2$.
4. $a^2+b^2=1$.
5. $y(x^2-1)=2$.
6. $(x^2-y^2)^2+16xy=0$.
7. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
8. $xy^2=4a^3$.
9. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
10. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}$.
11. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.
12. $\dfrac{x^2}{a}+\dfrac{y^2}{b}=a+b$.
13. $\dfrac{b^2}{a^2}-\dfrac{bd}{ac}=2$.
14. $\cot\alpha=\dfrac{1}{a}-\dfrac{1}{b}$.
15. $ab(ab-4)=(a+b)^2\tan^2\alpha$.
16. $8bc=a\{4b^2+(b^2-c^2)^2\}$.
17. $\tan^2\alpha=\tan^2\beta+\tan^2\gamma$.

18. $2(a^2x^2+b^2y^2)^3=(a^2-b^2)^3(a^2x^2-b^2y^2)^2$.

19. $(a^2+b^2)(c-1)+2b(c+1)=0$.

20. $\dfrac{a(m+b)}{\sqrt{(n+b)^2+(m+b)^2}}=\dfrac{mn-b^2}{n+b}$.

21. $(x-y)^{\frac{2}{3}}+(x+y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$.

22. $m^{\frac{2}{3}}+n^{\frac{2}{3}}=(mn)^{-\frac{2}{3}}$.

23. $(x\cos\alpha+y\sin\alpha)^{\frac{2}{3}}+(x\sin\alpha-y\cos\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$.

24. $a^2+c^2-2ac\cos 2\phi=b^2$.

25. $a(x^2+y^2)+2a^3+6a^2x=\pm(3a^2+2ax)^{\frac{3}{2}}$.

Examples XLIV. (page 313).

1. $\cdot 0014544$, $\cdot 9999989$.
2. $\frac{1}{15}$ radian.
3. $\cdot 0008727$, $\cdot 9999996$.
4. $\cdot 0040891$ radian.
5. $\frac{1}{29}$ radian.
6. $\frac{1}{17}$ radian.
7. $\cdot 011547$ radian.
8. $\frac{1}{37}$ radian.
9. $-\cdot 000513$ radian.
10. $\frac{1}{21}$ radian.
11. $3'\cdot 72$.
12. $\cdot 86353$.
13. $13\cdot 06$ metres.
14. $15\cdot 8$ metres.

Examples XLV. (page 320).

1. $\dfrac{\sin\frac{3n+1}{2}A\sin\frac{3nA}{2}}{\sin\frac{3A}{2}}$.

2. $\dfrac{\cos nA\sin nA}{\sin A}$.

3. $\dfrac{\cos\frac{3n-1}{6}A\sin\frac{nA}{2}}{\sin\frac{A}{2}}$.

4. $\dfrac{\cos\left\{\theta+\frac{(n-1)\pi}{2n}\right\}}{\sin\frac{\pi}{2n}}$.

5. 0.

6. $\dfrac{\sin\frac{n+1}{4}A\sin\frac{n}{4}A}{\sin\frac{A}{4}}$.

7. $\dfrac{\sin^2 \dfrac{10\pi}{21}}{\sin \dfrac{\pi}{21}}$.

8. $\dfrac{\cos \dfrac{11\pi}{23} \sin \dfrac{11\pi}{23}}{\sin \dfrac{\pi}{23}} - \dfrac{1}{2}$.

9. $\dfrac{\sin^2 \dfrac{n\pi}{2n-1}}{\sin \dfrac{\pi}{2n-1}}$.

10. $\dfrac{\sin\left(\dfrac{n+1}{2}\alpha + \dfrac{n-1}{2}\pi\right) \sin \dfrac{n(\alpha+\pi)}{2}}{\sin \dfrac{\alpha+\pi}{2}}$.

11. $\dfrac{\cos\left\{(n+1)\alpha + \dfrac{n-1}{2}\pi\right\} \sin \dfrac{n(2\alpha+\pi)}{2}}{\sin \dfrac{2\alpha+\pi}{2}}$.

12. $\dfrac{\sin\left(2\alpha + \dfrac{n^2-1}{2n}\pi\right) \sin \dfrac{n+1}{2}\pi}{\sin \dfrac{n+1}{2n}\pi}$.

13. $\dfrac{\cos\left\{3\alpha + \dfrac{(n-1)^2}{2n}\pi\right\} \sin \dfrac{n-1}{2}\pi}{\sin \dfrac{n-1}{2n}\pi}$.

14. $\dfrac{n}{2}\cos 2\alpha - \dfrac{\cos(n+1)2\alpha \sin 2n\alpha}{2\sin 2\alpha}$.

15. $\dfrac{n}{2}\cos 2\alpha + \dfrac{\cos(n+1)2\alpha \sin 2n\alpha}{2\sin 2\alpha}$.

16. $\dfrac{\sin(2n+3)\theta \sin 2n\theta}{2\sin 2\theta} - \dfrac{n\sin 3\theta}{2}$.

17. $\dfrac{\tan(2n+1)\alpha - \tan\alpha}{\sin 2\alpha}$.

18. $\dfrac{\cot\alpha - \cot(3n+1)\alpha}{\sin 3\alpha}$.

19. $\dfrac{\tan 2(n+1)\alpha - \tan 2\alpha}{\sin 2\alpha}$.

20. $\dfrac{\cot 2\alpha - \cot (n+2)\alpha}{\sin \alpha}$.

21. $\dfrac{n}{2} - \dfrac{\cos (2\alpha + \overline{n-1}\,\beta) \sin n\beta}{2 \sin \beta}$.

22. $\dfrac{n}{2} + \dfrac{\cos (n+3)\alpha \sin n\alpha}{2 \sin \alpha}$.

23. $\dfrac{n}{2}$.

24. $$\frac{3 \cos \left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{4 \sin \frac{\beta}{2}} + \frac{\cos 3\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{3n\beta}{2}}{4 \sin \frac{3\beta}{2}}.$$

25. $$\frac{3 \sin \frac{n+1}{2}\alpha \sin \frac{n\alpha}{2}}{4 \sin \frac{\alpha}{2}} - \frac{\sin \frac{3(n+1)}{2}\alpha \sin \frac{3n\alpha}{2}}{4 \sin \frac{3\alpha}{2}}.$$

26. $\frac{1}{8}[3n - 4 \cos (n+1)\alpha \sin n\alpha \operatorname{cosec} \alpha$
$+ \cos (2n+2)\alpha \sin 2n\alpha \operatorname{cosec} 2\alpha]$.

27. $$\frac{1}{8}\Big[3n + 4 \cos 2n\alpha \sin 2n\alpha \operatorname{cosec} 2\alpha + \cos 4n\alpha \sin 4n\alpha \operatorname{cosec} 4\alpha\Big].$$

28. $-\sin \dfrac{n\alpha}{n-2}$.

29. $\frac{1}{2} \operatorname{cosec} \theta \{\tan (n+1)\theta - \tan \theta\}$.

30. $$\frac{\sin \left(\frac{n+1}{2}\alpha + \frac{n-1}{2}\pi\right) \sin \frac{n(\pi+\alpha)}{2}}{\sin \frac{\pi+\alpha}{2}}.$$

31. $$\frac{1}{4} \sin \frac{n\alpha}{2}\left[\cos \frac{n-1}{2}\alpha + \cos \frac{n+3}{2}\alpha + \cos \frac{n+7}{2}\alpha\right] \operatorname{cosec} \frac{\alpha}{2} + \frac{1}{4} \sin \frac{3n\alpha}{2} \cos \frac{3n+9}{2}\alpha \operatorname{cosec} \frac{3\alpha}{2}.$$

32. $\tan^{-1}(n+1)x - \tan^{-1} x$.

33. $\tan^{-1}(n+1) - \dfrac{\pi}{4}$.

Examples XLVII. (page 340).

1. $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

2. $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

3. $2\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right\}$.

4. $5\{\cos(\tan^{-1}\frac{3}{4})+i\sin(\tan^{-1}\frac{3}{4})\}$

or $$5\{\cos 36^\circ 52'+i\sin 36^\circ 52'\}.$$

5. $\sqrt{298}\{\cos(\tan^{-1}\frac{17}{3})+i\sin(\tan^{-1}\frac{17}{3})\}$

or $$\sqrt{298}\{\cos 80^\circ+i\sin 80^\circ\}.$$

6. $2^{\frac{1}{4}}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$; $2^{\frac{1}{4}}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$;

$2^{\frac{1}{4}}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$; $2^{\frac{1}{4}}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$.

7. $\sqrt{29}(\cos 21^\circ 48'+i\sin 21^\circ 48')$;
$\sqrt{29}(\cos 141^\circ 48'+i\sin 141^\circ 48')$;
$\sqrt{29}(\cos 261^\circ 48'+i\sin 261^\circ 48')$.

8. $k\left(\cos\frac{5\pi}{72}+i\sin\frac{5\pi}{72}\right)$; $k\left(\cos\frac{29\pi}{72}+i\sin\frac{29\pi}{72}\right)$;

$k\left(\cos\frac{53\pi}{72}+i\sin\frac{53\pi}{72}\right)$; $k\left(\cos\frac{77\pi}{72}+i\sin\frac{77\pi}{72}\right)$;

$k\left(\cos\frac{101\pi}{72}+i\sin\frac{101\pi}{72}\right)$; $k\left(\cos\frac{125\pi}{72}+i\sin\frac{125\pi}{72}\right)$;

where $$k=(\sqrt{6}-\sqrt{2})^{\frac{1}{6}}.$$

9. $\cos(8\theta-9\phi)+i\sin(8\theta-9\phi)$.

10. $\cos(9\theta+7\phi)-i\sin(9\theta+7\phi)$.

11. $\cos 16\theta+i\sin 16\theta$.

12. $\cos 20\theta+i\sin 20\theta$.

Examples XLVIII. (page 347).

1. ·8414710
·5403023.

2. $\cos\alpha - h\sin\alpha - \dfrac{h^2}{|\underline{2}}\cos\alpha + \dfrac{h^3}{|\underline{3}}\sin\alpha + \dots$

3. $(-1)^n \dfrac{3^{2n}+3}{4|\underline{2n}}\theta^{2n}$.

4. $\theta = \frac{1}{21}$ radian.

5. $(-1)^n \dfrac{2^{2n-1}(1-2^{2n})}{|\underline{2n+1}}\theta^{2n+1}$.

6. $\cos\alpha$.

7. $-\frac{16}{15}$.

8. $-\frac{1}{30}$.

9. $\frac{5}{3}$.

10. $-\dfrac{a^2+ab+b^2}{ab}$.

11. 2.

18. $1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4 + \frac{61}{720}\theta^6 + \frac{277}{8064}\theta^8$.

TEST PAPERS.

XLVI. (page 350).

6. $\tan A \tan B$ or q.

XLVII. (page 351).

3. 2297 yards.

4. 2·903 metres.

5. 7575 sq. metres.

XLVIII. (page 352).

5. $B = 99° 35'$; $C = 55° 24'$; $b = 4997$,

or $B = 30° 23'$; $C = 124° 36'$; $b = 2564$.

6. 6087 ft.

XLIX. (page 353).

1. $B = 37° 18'$; $C = 83° 1'$; $a = 11060$.

7. $120°$.

LII. (page 355).

1. 1·152 miles.

2. $\frac{4n-1}{2}\pi$; $\frac{4n+1}{10}\pi$.

6. $n\pi$; $n\pi \pm 26° 34'$.

LIII. (page 356).

1. $(2n+1)\pi$, $n\pi \pm \frac{\pi}{6}$.

3. 1730 metres.

LIV. (page 357).

2. (i) $\frac{n\pi}{2} + \frac{\pi}{8}$.

   (ii) $\frac{n\pi}{4} + (-1)^n \frac{\pi}{12}$.

4. 50·1 metres.

5. $15° 38'$.

6. 39740 sq. cms.

LV. (page 358).

1. $x = \frac{3}{2}$.

3. $\frac{n\pi}{3} + 8° 51'$.

LVI. (page 359).

2. (i) $\pm ·1602$; $\pm 6·2432$.

   (ii) $\frac{2n+1 \pm \sqrt{4n^2+4n-15}}{4}$.

4. $\frac{-1 \pm \sqrt{2+b^2}}{a}$.

MISCELLANEOUS EXAMPLES (page 361).

7. $\theta = (2n-1)\frac{\pi}{2}, \frac{4n \pm 1}{11}\pi, \frac{4n-1}{6}\pi$.

24. $\theta = (2n+1)\frac{\pi}{2}, \frac{4n \pm 3}{2}\pi$.

30. $\frac{12}{35}$.

40. $n\pi \pm \frac{\pi}{4}, 2n\pi \pm \frac{\pi}{3}$.

52. $(2n+1)\frac{\pi}{10}, (2n+1)\frac{\pi}{2}$.

56. $2n\pi + 2\tan^{-1}\frac{b \pm \sqrt{a^2+b^2-c^2}}{a+c}$.

59. 45°.

66. 869·2575 feet, 1038023 sq. ft.

67. (i) $2m\pi, \frac{2}{3}\left(m\pi - \tan^{-1}\frac{b}{a}\right)$; (ii) $\frac{4n+1}{10}\pi, \frac{4n \pm 1}{2}\pi$.

71. 2954·4 feet.

78. $2m\pi - \alpha, \frac{4n-1}{2}\pi + \alpha$.

95. $(2n+1)\frac{\pi}{6}, \frac{2m\pi + \alpha + \gamma + \beta}{3}$.

102. $(a^2+b^2)(a^2+b^2-3) - 2b = 0$.

120. $a = \cdot 835$.

131. $= 11370$.

EXAMPLES XLIX. (page 383).

1. $9{\cdot}123210 \times 10^5$.
2. $3{\cdot}198466 \times 10^{-5}$.
3. $3{\cdot}119093 \times 10^5$.
4. $9{\cdot}013538 \times 10^{-8}$.
5. $1{\cdot}0623230 \times 10^{-5}$.
6. $4{\cdot}107103 \times 10$.
7. $1{\cdot}287775$.
8. $6{\cdot}021347 \times 10^{-4}$.
9. $5{\cdot}129823 \times 10^5$.
10. $6{\cdot}699801 \times 10^{-8}$.
11. $3{\cdot}011193 \times 10^{-5}$.
12. $8{\cdot}17765 \times 10^{-4}$.
13. $1{\cdot}072324 \times 10$.
14. $4{\cdot}700017 \times 10^{-8}$.
15. $1{\cdot}200537 \times 10^{-5}$.

EXAMPLES L. (page 393).

1. 63·1. 2. 245·6. 3. ·000314.

4. 19·9. 5. 681. 6. 3·38.

7. 1877. 8. 7·5. 9. 82·5.

10. 315. 11. 55. 12. ·341.

13. 107000. 14. 467. 15. 105·5.

16. 3·42. 17. 119. 18. 1·619. 19. ·0889.

20. (i) $3·8 \times 10^3$. (ii) $2·78 \times 10^{-3}$. (iii) $3·438 \times 10^4$. (iv) $1·06 \times 10^5$. (v) $3·4 \times 10^{-6}$.

21. (i) 9·25. (ii) $1·02 \times 10$. (iii) $2·69 \times 10^{-1}$. (iv) $9·46 \times 10^{-3}$. (v) $9·08 \times 10$.

22. (i) $4·36 \times 10^5$. (ii) $5·54 \times 10^8$. (iii) $4·288 \times 10^{-3}$. (iv) $2·05 \times 10^{-7}$.

23. (i) 4·18. (ii) 9·38. (iii) 1·998. (iv) $1·8 \times 10^{-1}$. (v) $9·225 \times 10^{-2}$.

EXAMPLES LI. (page 400).

1. 412·4°. 2. 1·082 radians.

3. 106 sq. cms. 4. 45·2 cms.

5. 9640 cu. dms. 6. 194·7".

7. 22 cms. 8. 201 sq. cms.

9. 2·44 radians. 10. 16·43 cms.

11. 16850 cu. cms. 12. 17·84 cms.

13. 125·7 sq. cms. 14. B = 105° 45′, C = 24° 15′.

15. 1331 kiloms. 16. 113600 kilogs.

17. 805000 cms. 18. C = 62° 48′, A = 42° 22′.

19. 735·2 sq. cms. 20. 3960 miles.

21. $6·04 \times 10^{21}$ tons.

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